Survey of Systems of Strength $\psi(\Gamma_{\Omega+1})$

Ulrik Torben Buchholtz
jww Gerhard Jäger and Thomas Strahm

University of Bern

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Outline

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Reminder on predicativity

- Russell and Poincaré worried about *vicious circles* as a source of paradoxes. Thus *predicativity* emerged as a prescription to avoid paradoxes by avoiding vicious circles.
- No universal agreement on what exactly that means, though!
- Standard analysis takes the set of natural numbers as given. On the Feferman-Schütte analysis, the ordinal Γ_0 arises as the limit of predicatively acceptable ordinals, and as the proof-theoretic ordinal of various limits of predicatively acceptable systems.
- $\Gamma_0 := \lim_n \xi_n$ where $\xi_0 := \varepsilon_0 := \lim_n \omega^{\cdot^{\cdot^{\cdot^{\omega^0}}}}$ (tower with n ωs), and $\xi_{n+1} := \varphi(\xi_n, 0)$ where φ is the binary Veblen function obtained by enumerating fixed points starting with the normal function $\lambda\alpha.\,\omega^\alpha$.

Predicative closure seems ubiquitous

All the following systems have proof-theoretic ordinal Γ_0:

- Aut(RA) (autonomous closure of ramified analysis),
- $\mathscr{U}(\mathsf{NFA})$ (Feferman's unfolding of non-finitist arithmetic),
- ATR_0 (H. Friedman's system of arithmetic transfinite recursion),
- FP_0 (Avigad's fixed point theory),
- IR (Feferman's system of Induction-Recusion),
- Δ^1_1-$\mathsf{DC}_0 + (\mathsf{SUB})$ (Δ^1_1 dependent choices plus substitution in second order arithmetic),
- $\widehat{\mathsf{ID}}_{<\omega}$ (finitely many generalized arithmetic fixed points),
- $\mathsf{ML}_{<\omega}$ (predicative Martin-Löf type theory with a hierarchy of universes).
- a system of Explicit Mathematics with a hierarchy of universes *with induction restricted to types* (otherwise the strength jumps to that of $\widehat{\mathsf{ID}}_{<\varepsilon_0}$),
- . . . .

Analogous systems for other notions of predicativity?

- If we do not have the natural numbers as a given, get a notion of predicative closure of finitism, $\mathscr{U}(\mathsf{FA})$, equivalent to primitive recursive arithmetic (Feferman and Strahm).
- With a stricter notion of quantification, we obtain a predicative closure of feasible arithmetic, $\mathscr{U}(\mathsf{FEA})$, with provably total functions exactly the polynomial time computable ones (Eberhard and Strahm).
- How about stronger notions? What happens if we take one generalized (arithmetic) inductive definition as given?

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Unfolding of ID_1

Feferman proposed to look at $\mathscr{U}(\mathrm{ID}_1)$, as another example of predicative closure.

Formal system for one arithmetical inductive definition, ID_1

The language of ID_1, $\mathscr{L}^1$, is that of first-order arithmetic, PA, (with a free predicative variable U) augmented by a predicate symbol $I_{\mathscr{A}}$ for an arithmetical positive operator form $\mathscr{A}(X,x)$ that does not contain U.

The universal case is that of Kleene's $\mathscr{O}$ consisting of ordinal notations for recursive ordinals.

The unfoldings are defined for *schematic systems* in first-order logic.

Schematic systems, S, are formulated with a free predicate variable U accompanied by a rule of formula substitution

(Subst) $$\frac{A(U)}{A(\{x \mid B(x)\})}$$

Examples of schematic systems

- Non-finitist arithmetic, NFA with

$$U(0) \wedge (\forall x.\, U(x) \to U(x')) \to \forall x.\, U(x)$$

- Zermelo's set theory with

$$\forall a.\, \exists b.\, \forall x.\, x \in b \leftrightarrow x \in a \wedge U(x).$$

Operational unfolding

- The universe of the operational unfolding consists of the original sorts of S, embedding into a single sort of operations by means of predicates V_s.
- Each n-ary function symbol f of S determines an total n-ary operation f^* on the corresponding sorts.
- Machinery to define new operations by recursion and explicit definition.
- Each n-ary predicate symbol R of S determines a predicate R^*.
- The axioms of S are included, relativized to the V_s.
- The logic of $\mathcal{U}_0(\mathsf{S})$ is the *logic of partial terms*.

We include in $\mathscr{U}_0(\mathsf{S})$ the substitution rule:

$$\frac{A(U)}{A(\{\, x \mid B(x) \,\})}\ \text{(Subst)}$$

- The language of $\mathscr{U}(\mathsf{S})$ extends the language of $\mathscr{U}_0(\mathsf{S})$ by additional constants for the predicate symbols of S plus $\mathrm{Eq}, \mathrm{Pr}_U, \mathrm{Inv}, \mathrm{Neg}, \mathrm{Conj}, \mathrm{Un}, \mathrm{Join}$.

- $\mathrm{Eq}\downarrow \land \forall x,y.\,(x,y) \in \mathrm{Eq} \leftrightarrow x = y.$
- $\mathrm{Pr}_U\downarrow \land \forall x.x \in \mathrm{Pr}_U \leftrightarrow U(x).$
- $\mathrm{Inv}(X,f_1,\ldots,f_m)\downarrow \land$
  $\forall \vec{x}.\vec{x} \in \mathrm{Inv}(X,f_1,\ldots,f_m) \leftrightarrow (f_1(\vec{x}),\ldots,f_m(\vec{x})) \in X.$
- $\mathrm{Neg}(X)\downarrow \land \forall \vec{x}.\vec{x} \in \mathrm{Neg}(X) \leftrightarrow \vec{x} \notin X.$
- $\mathrm{Conj}(X,Y)\downarrow \land \forall \vec{x}.\vec{x} \in \mathrm{Conj}(X,Y) \leftrightarrow \vec{x} \in X \land \vec{x} \in Y.$
- $\mathrm{Un}(X)\downarrow \land \forall \vec{x}.\vec{x} \in \mathrm{Un}(X) \leftrightarrow \forall y.\,(\vec{x},y) \in X.$
- For $f : \iota \rightharpoonup \pi_n$ and $r : \pi_1$ we take

$$(\forall y.y \in r \to f(y)\downarrow) \to \mathrm{Join}(f,r)\downarrow \land$$
$$\forall \vec{x},y.\,(\vec{x},y) \in \mathrm{Join}(f,r) \leftrightarrow y \in r \land \vec{x} \in f(y).$$

For $\mathscr{U}(\mathsf{S})$ we restrict the substitution rule,

$$\frac{A(U)}{A(\{\, x \mid B(x) \,\})}\ \text{(Subst)}\ ,$$

by requiring A to be in the language of $\mathscr{U}_0(\mathsf{S})$. This is needed, because the full unfolding language reflects the free predicate U.

Theorem

$\mathcal{U}(\mathsf{ID}_1)$ *has proof-theoretic strength* $\psi(\Gamma_{\Omega+1})$.

Remark

$\psi(\Gamma_{\Omega+1})$ figured in the original 1950 paper of Bachmann that inspired Howard's work on ID_1.

Definition

By recursion on α we define simultaneously

$$\begin{aligned}\mathrm{Cl}(\alpha,\beta) := {} & \text{the least set containing } \beta \cup \{0,\Omega\} \\ & \text{and closed under } +\text{, the Veblen function } \lambda\xi\eta.\,\varphi_\xi(\eta), \\ & \text{and the restricted function } \psi \restriction \alpha := \lambda\xi < \alpha.\,\Psi(\xi), \\ \psi(\alpha) := {} & \min\{\beta \mid \mathrm{Cl}(\alpha,\beta) \cap \Omega \subseteq \beta\}.\end{aligned}$$

Thus, $\psi(\Gamma_{\Omega+1})$ is the collapse of the first strongly-critical ordinal greater than Ω.

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We can model one generalized inductive definition in second order arithmetic as follows:

- Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be the languages of first and second order arithmetic, respectively.
- Let $\mathfrak{A}[P, u]$ range over P-*positive* formulas of $\mathscr{L}_1(P)$ that contains at most u free (these are the *inductive operator forms*).
- Extend $\mathscr{L}_2$ to $\mathscr{L}_2^\bullet$ by added a fresh unary relation symbol $P_\mathfrak{A}$ for each inductive operator form $\mathfrak{A}[P, u]$.
- An $\mathscr{L}_2^\bullet$ formula is called *elementary* in case it does not contain bounded set variables.

- Recall the schema of *arithmetic comprehension*:

$$\exists X.\, \forall a.\, a \in X \leftrightarrow A[a],$$

for arithmetic formulas $A[u]$ of $\mathscr{L}_2$.

- Recall also the *induction axiom*:

$$\forall X.\, 0 \in X \land (\forall a.\, a \in X \to a' \in X) \to \forall a.\, a \in X.$$

- These define the system ACA_0.
- Recall also the comprehension and choice principles:

(Δ^1_1-CA)
$$(\forall a.\, (\exists X.\, A[X,a]) \leftrightarrow \forall X.\, B[X,a]) \to \exists Y.\, \forall a.\, a \in Y \leftrightarrow \exists X.\, A[X,a],$$

(Σ^1_1-AC) $\quad (\forall a.\, \exists X.\, C[a,X]) \to \exists Y.\, \forall a.\, C[a,(Y)_a],$

(Σ^1_1-DC) $\quad (\forall a.\, \forall X.\, \exists Y.\, D[a,X,Y]) \to \exists Z.\, \forall a.\, D[a,(Z)^a,(Z)_a],$

- The *substitution rule* is the rule of inference:

(SUB) $$\frac{\forall X. A[X]}{A[\{a : B[a]\}]}$$

for *arithmetic* $A[X]$ and arbitrary $B[v]$.

- Recall a theorem of Avigad: the theory ATR_0 is equivalent over $\mathscr{L}_2$ to the theory FP_0, which is defined as ACA_0 plus the fixed point schema:

(FP) $$\exists X. \forall a. a \in X \leftrightarrow A[X, a],$$

for U-positive arithmetic formulas $A[U, v]$.

- Now we extend these theories to language $\mathscr{L}_2^\bullet$ and add the *least fixed point* axioms:

$$\text{(ID.1)} \qquad \forall a.\, \mathfrak{A}[P_{\mathfrak{A}}, a] \to P_{\mathfrak{A}}(a),$$

$$\text{(ID.2)} \qquad \forall X.\, (\forall a.\, \mathfrak{A}[X, a] \to a \in X) \to \forall a.\, P_{\mathfrak{A}}(a) \to a \in X.$$

- We also add the *elementary comprehension axiom*:

$$\text{(E-CA)} \qquad \exists X.\, \forall a.\, a \in X \leftrightarrow A[a],$$

  for elementary formulas $A[u]$. We get the theory $\mathsf{ACA}_0^\bullet$, which conservatively extends the first-order theory ID_1.

- We get theories Δ^1_1-$\mathsf{CA}_0^\bullet$, Σ^1_1-$\mathsf{AC}_0^\bullet$ and Σ^1_1-$\mathsf{DC}_0^\bullet$ by adding the corresponding schemata with arithmetic replaced with elementary.

Theorem

The following theories all have proof-theoretic ordinal $\psi(\Gamma_{\Omega+1})$*:*

- Δ^1_1-$\mathsf{CA}^\bullet_0 + \mathsf{SUB}^\bullet$,
- Σ^1_1-$\mathsf{AC}^\bullet_0 + \mathsf{SUB}^\bullet$,
- Σ^1_1-$\mathsf{DC}^\bullet_0 + \mathsf{SUB}^\bullet$,
- $\mathsf{FP}^\bullet_0$,
- $\mathsf{ATR}^\bullet_0$,
- $\mathsf{ATR}_0 + (\mathsf{LFP})$, *where* (LFP) *is the schema*

  $$(\mathsf{LFP})\quad (\exists X.\,\forall a.\,\mathfrak{A}[X,a] \to X(a)) \wedge \forall Y.\,(\forall a.\,\mathfrak{A}[Y,a] \to a \in Y) \to X \subseteq Y,$$

  for all inductive operator forms $\mathfrak{A}[U,v]$*.*

In fact, we have equivalence for elementary Π^1_1-sentences.

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$$\frac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \mathscr{T} : \mathscr{U}_i}$$

$$\frac{\Gamma\ \mathsf{ctx}}{0 : \mathscr{T}} \qquad \frac{\Gamma \vdash a : \mathscr{T}}{\Gamma \vdash \mathrm{suc}(a) : \mathscr{T}} \qquad \frac{\Gamma \vdash f : \mathbb{N} \to \mathscr{T}}{\Gamma \vdash \lim(f) : \mathscr{T}}$$

$$\frac{\begin{array}{cc} & \Gamma \vdash c : C[0/z] \\ \Gamma \vdash a : \mathscr{T} & \Gamma, x : \mathscr{T}, y : C[x/z] \vdash d : C[\mathrm{suc}(x)/z] \\ \Gamma, z : \mathscr{T} \vdash C : \mathscr{U}_i & \Gamma, x : \mathbb{N} \to \mathscr{T}, y : \Pi_{n:\mathbb{N}} C[x(n)/z] \vdash e : C[\lim(x)/z] \end{array}}{\Gamma \vdash \mathrm{ind}_{\mathscr{T}}(z.C, c, xy.d, xy.e, a) : C[a/z]}$$

$$\frac{\Gamma, z : \mathscr{T} \vdash C : \mathscr{U}_i \qquad \begin{array}{c} \Gamma \vdash c : C[0/z] \\ \Gamma, x : \mathscr{T}, y : C[x/z] \vdash d : C[\mathrm{suc}(x)/z] \\ \Gamma, x : \mathbb{N} \to \mathscr{T}, y : \Pi_{n:\mathbb{N}} C[x(n)/z] \vdash e : C[\lim(x)/z] \end{array}}{\Gamma \vdash \mathrm{ind}_{\mathscr{T}}(z.C, c, xy.d, xy.e, 0) \equiv c : C[0/z]}$$

$$\frac{\begin{array}{c} \Gamma \vdash a : \mathscr{T} \\ \Gamma, z : \mathscr{T} \vdash C : \mathscr{U}_i \end{array} \qquad \begin{array}{c} \Gamma \vdash c : C[0/z] \\ \Gamma, x : \mathscr{T}, y : C[x/z] \vdash d : C[\mathrm{suc}(x)/z] \\ \Gamma, x : \mathbb{N} \to \mathscr{T}, y : \Pi_{n:\mathbb{N}} C[x(n)/z] \vdash e : C[\lim(x)/z] \end{array}}{\begin{array}{c} \Gamma \vdash \mathrm{ind}_{\mathscr{T}}(z.C, c, xy.d, xy.e, \mathrm{suc}(a)) \\ \equiv d[a, \mathrm{ind}_{\mathscr{T}}(z.C, c, xy.d, xy.e, a)/x, y] : C[\mathrm{suc}(a)/z] \end{array}}$$

$$\frac{\begin{array}{c} \Gamma \vdash f : \mathbb{N} \to \mathscr{T} \\ \Gamma, z : \mathscr{T} \vdash C : \mathscr{U}_i \end{array} \qquad \begin{array}{c} \Gamma \vdash c : C[0/z] \\ \Gamma, x : \mathscr{T}, y : C[x/z] \vdash d : C[\mathrm{suc}(x)/z] \\ \Gamma, x : \mathbb{N} \to \mathscr{T}, y : \Pi_{n:\mathbb{N}} C[x(n)/z] \vdash e : C[\lim(x)/z] \end{array}}{\begin{array}{c} \Gamma \vdash \mathrm{ind}_{\mathscr{T}}(z.C, c, xy.d, xy.e, \lim(f)) \\ \equiv e[f, \lambda n : \mathbb{N}.\, \mathrm{ind}_{\mathscr{T}}(z.C, c, xy.d, xy.e, f(n))/x, y] : C[\lim(f)/z] \end{array}}$$

Outline

UCN$\mathcal{T}$

Here we introduce a system, UCN$\mathcal{T}$, in the style of Explicit Mathematics with a type of tree ordinals, $\mathcal{T}$, and a hierarchy of universes.
UCN$\mathcal{T}$ is, as usually, formulated in classical two-sorted (operations and classes) first-order logic, where for the operation we have *partial terms*.
The operational sort includes

- variables $a, b, c, f, g, h, u, v, w, x, y, z, \ldots$
- constants k, s (combinators), $\mathrm{p}, \mathrm{p}_0, \mathrm{p}_1$ (pairing and projections), 0 (zero), $\mathrm{s_N}$ (numerical successor), $\mathrm{p_N}$ (numerical predecessor), $\mathrm{d_N}$ (definition by numerical cases), $\mathrm{r_N}$ (primitive recursion), . . . (we will introduce more later)

UCN$\mathscr{T}$, continued

The individual terms ($r, s, t, \ldots$) are inductively generated from variables and constants by application terms $\cdot(s, t)$, also written st (left-associative).

We write $s(t_1, \ldots, t_n)$ for $st_1 \ldots t_n$, and $\langle s, t \rangle$ for $\mathrm{p}(s, t)$.

$$(s \simeq t) := (s\downarrow \vee\, t\downarrow) \rightarrow (s = t)$$

- Partial combinatory algebra.
  1. $\mathrm{k}\,a\,b = a$.
  2. $\mathrm{s}\,a\,b{\downarrow} \wedge \mathrm{s}\,a\,b\,c \simeq a\,c\,(b\,c)$.
- Pairing and projection.
  1. $\mathrm{p}_0\langle a,b\rangle = a \wedge \mathrm{p}_1\langle a,b\rangle = b$.
- Natural numbers.
  1. $0 \in \mathsf{N} \wedge (a \in \mathsf{N} \to a' \in \mathsf{N})$.
  2. $a \in \mathsf{N} \to a' \neq 0 \wedge \mathrm{p}_{\mathsf{N}}(a') = a$.
  3. $a \in \mathsf{N} \wedge a \neq 0 \to \mathrm{p}_{\mathsf{N}}a \in \mathsf{N} \wedge (\mathrm{p}_{\mathsf{N}}a)' = a$.
- Definition by cases on N.
  1. $a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge a = b \to \mathrm{d}_{\mathsf{N}}\,u\,v\,a\,b = u$.
  2. $a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge a \neq b \to \mathrm{d}_{\mathsf{N}}\,u\,v\,a\,b = v$.
- Primitive recursion on N.
  1. $f \in (\mathsf{N}^2 \to \mathsf{N}) \wedge a \in \mathsf{N} \to \mathrm{r}_{\mathsf{N}}\,f\,a \in (\mathsf{N} \to \mathsf{N})$.
  2. $f \in (\mathsf{N}^2 \to \mathsf{N}) \wedge a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge h = \mathrm{r}_{\mathsf{N}}\,f\,a \to$
     $h\,0 = a \wedge h(b') = f\,b\,(h\,b)$.

We have additional operational constants: nat (natural numbers), id (identity), co (complement), un (union), dom (domain), inv (inverse image). There are two new relation symbols $\in$ (element) and $\mathfrak{R}$ (naming).

- Explicit representation and extensionality.
  1. $\exists x.\, \mathfrak{R}(x, U)$.
  2. $\mathfrak{R}(s, U) \wedge \mathfrak{R}(s, v) \rightarrow U = V$.
  3. $(\forall x. x \in U \leftrightarrow x \in V) \rightarrow U = V$.
- Class existence.
  1. $\mathfrak{R}(\mathrm{nat}) \wedge \forall x. x \mathrel{\dot\in} \mathrm{nat} \leftrightarrow x \in \mathsf{N}$.
  2. $\mathfrak{R}(\mathrm{id}) \wedge \forall x. x \mathrel{\dot\in} \mathrm{id} \leftrightarrow \exists y. x = \langle y, y \rangle$.
  3. $\mathfrak{R}(s) \rightarrow \mathfrak{R}(\mathrm{co}(s)) \wedge \forall x. x \mathrel{\dot\in} \mathrm{co}(s) \leftrightarrow x \mathrel{\dot\notin} s$.
  4. $\mathfrak{R}(s) \wedge \mathfrak{R}(t) \rightarrow \mathfrak{R}(\mathrm{un}(s, t)) \wedge \forall x. x \mathrel{\dot\in} \mathrm{un}(s, t) \leftrightarrow x \mathrel{\dot\in} s \vee x \mathrel{\dot\in} t$.
  5. $\mathfrak{R}(s) \rightarrow \mathfrak{R}(\mathrm{dom}(s)) \wedge \forall x. x \mathrel{\dot\in} \mathrm{dom}(s) \leftrightarrow \exists y.\, \langle x, y \rangle \mathrel{\dot\in} s$.
  6. $\mathfrak{R}(s) \rightarrow \mathfrak{R}(\mathrm{inv}(s, f)) \wedge \forall x. x \mathrel{\dot\in} \mathrm{inv}(s, f) \leftrightarrow f\, x \mathrel{\dot\in} s$.
- Class induction.

$$\forall X.\, 0 \in X \wedge (\forall x \in \mathsf{N}. x \in X \rightarrow x' \in X) \rightarrow \forall x \in \mathsf{N}. x \in X.$$

Join and Universes

- Join adds a constant j such that

$$\mathfrak{R}(a) \wedge (\forall x \mathrel{\dot\in} a.\, \mathfrak{R}(f\,x)) \to \mathfrak{R}(\mathsf{j}(a,f) \wedge \\ \forall z.\, z \mathrel{\dot\in} \mathsf{j}(a,f) \leftrightarrow \exists xy.\, z = \langle x,y \rangle \wedge x \mathrel{\dot\in} a \wedge y \mathrel{\dot\in} f\,x.$$

- Universes add a relation symbol U such that
  1. $a \in \mathsf{U} \wedge b \mathrel{\dot\in} a \to \mathfrak{R}(a)$.
  2. $a \in \mathsf{U} \to \mathrm{nat} \mathrel{\dot\in} a$.
  3. Universes are closed under elementary comprehension and join.
  4. $\mathfrak{R}(a) \to \exists u \in \mathsf{U}.\, a \mathrel{\dot\in} u$.
  5. Universes are linearly ordered and cumulative.

Some earlier results

- $\mathsf{ECJ} + n$ universes $+ (\mathsf{C\text{-}I_N}) \equiv \widehat{\mathsf{ID}}_n$ (Feferman '82).
- $|\mathsf{UCN}| = \Gamma_0$ (Marzetta '93)

Tree ordinals

To obtain UCN$\mathscr{T}$ we add a name tree for a class $\mathscr{T}$, and constants s_T (tree successor), ℓ_T (tree limit), r_T (tree recursion). We have induction on $\mathscr{T}$ for classes.
All universes are now required to contain tree.

Theorem

The systems $\mathsf{ML}_{<\omega}\mathscr{T}$, $\mathsf{UCN}\mathscr{T}^{\mathsf{i}}$, $\mathsf{UCN}\mathscr{T}$ *all have proof-theoretic strength* $\psi(\Gamma_{\Omega+1})$.

Outline

Notes on the proofs

Lower bound

The lower bound is established with a well-ordering proof as in my thesis. This needs to be done for $\Delta^1_1\text{-CA}^\bullet_0 + (\text{SUB}^\bullet)$ and for the type theory (using techniques of Setzer and Hancock).

Reductions

$\Sigma^1_1\text{-DC}^\bullet_0 + (\text{SUB}^\bullet)$ reduces to $\text{ATR}_0 + (\text{LFP})$ using existence of enough countably coded ω-models of $\Sigma^1_1\text{-DC}_0$ in ATR_0.
All the mentioned theories then reduce to a theory $\widehat{\text{ID}}^\bullet_{<\omega}$ of fixed point theories on top of ID_1.

Upper bound

The theory $\widehat{\text{ID}}^\bullet_{<\omega}$ is handled via ordinal analysis.

Questions or Comments?