Proof-Theoretic Ordinals related to Unfoldings

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Outline

1. Background on ordinals
2. Ordinal function hierarchies
3. Relations to Unfoldings
   - Lower bound
   - Upper bound

Work with standard Von Neumann set-theoretic definition: an ordinal is a hereditarily transitive set. Then each ordinal is the *well-ordered set of all smaller ordinals.*

Every ordinal is either:

- the least ordinal, 0;
- a successor ordinal, i.e., an ordinal of the form $\alpha' = \alpha \cup \{\alpha\}$;
- a limit ordinal. The class of limit ordinals is denoted Lim.

Let Ω be the first uncountable ordinal. Then Ω is *regular*: if $M \subseteq \Omega$ is countable, then there exists some $\alpha < \Omega$ with $M \subseteq \alpha$.

Let ON denote the class of all ordinals. Every subclass M is then well-ordered with order-type $\leq$ ON. If M is bounded in ON, its order-type is an ordinal $\alpha \in$ ON and we have an enumeration function $\mathrm{en}_M : \alpha \to M$. If M is unbounded in ON, it has the order-type of ON itself and we have an enumeration function $\mathrm{en}_M : \mathrm{ON} \to M$.

A class $M \subseteq \mathrm{ON}$ is called *closed*, if M is closed in the order-topology, which is the case precisely when suprema of sets of elements in M themselves are in M.

Let $\kappa \in \{\Omega, \mathrm{ON}\}$. An ordinal function $f : \kappa \to \kappa$ is *continuous* when it preserves suprema. And f is called *normal* when it is strictly increasing and continuous. A normal function satisfies $f(\alpha) \geq \alpha$ for all α.

A class $M \subseteq \kappa$ is called κ-*club* if and only if it is *cl*osed and *u*n*b*ounded in κ. A fundamental fact is that M is club if and only its enumeration function en_M is normal.

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We let AP be the class of *additively principal* ordinals (so $\alpha \in \mathrm{AP}$ if and only if for all $\xi, \eta < \alpha$, $\xi + \eta < \alpha$). Then AP is κ-club for all $\kappa \in \{\Omega, \mathrm{ON}\}$ and its enumerating function is $\lambda\xi.\,\omega^\xi$.

By induction on $\alpha > 0$, we prove that there are uniquely determined ordinals $\{\,\alpha_1, \ldots, \alpha_n\,\} \subseteq \mathrm{AP}$ such that

$$\alpha = \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad \alpha_1 \geq \cdots \geq \alpha_n.$$

This is called the *additive normal form* of α, and $\{\,\alpha_1, \ldots, \alpha_n\,\}$ is called the set of *additive components* of α.

Using the enumerating function for AP we obtain the *Cantor normal form* for $\alpha > 0$:

$$\alpha =_{\mathrm{NF}} \omega^{\xi_1} + \cdots + \omega^{\xi_n} \quad \text{with} \quad \xi_1 \geq \cdots \geq \xi_n.$$

Any ordinal less than the first fixed point ε_0 of $\lambda\xi.\,\omega^\xi$ has a unique representation in hereditary Cantor normal form starting from 0.

For a class $M \subseteq \kappa$ we define its derivative as the class of the fixed points of its enumerating function:

$$M' := \{\, \xi < \kappa \mid \mathrm{en}_M(\xi) = \xi \,\}$$

The *derivative* f' of a function f is defined by $f' := \mathrm{en}_{\mathrm{Fix}(f)}$ where

$$\mathrm{Fix}(f) := \{\, \xi < \kappa \mid f(\xi) = \xi \,\}.$$

It is easily shown that if M is κ-club, then so is M'. Hence, if f is κ-normal, then so is f'.
If $\{\, M_\xi \,\}$ is a collection of less than κ many κ-club classes, then $\bigcap_\xi M_\xi$ is again κ-club.
Thus we can iterate the derivation process transfinitely to get a hierarchy of club classes. Starting with AP we obtain

$$\begin{aligned} \mathrm{Cr}(0) &:= \mathrm{AP} \\ \mathrm{Cr}(\alpha+1) &:= \mathrm{Cr}(\alpha)' \\ \mathrm{Cr}(\lambda) &:= \bigcap_{\xi<\lambda} \mathrm{Cr}(\xi) \quad \text{for } \lambda \in \mathrm{Lim}. \end{aligned}$$

Then we put $\varphi_\alpha := \mathrm{en}_{\mathrm{Cr}(\alpha)}$ to obtain the usual *Veblen hierarchy*.

Ordinals closed under the binary Veblen function, $\lambda\alpha, \beta.\, \varphi_\alpha(\beta)$, are called *strongly critical ordinals*. Their class, SC, is κ-club, and we let $\lambda\xi.\, \Gamma_\xi$ be their enumerating function.

The first strongly critical ordinal, Γ_0, is also known as the Feferman-Schütte ordinal, and is well-known from the analysis of predicativity given the natural numbers.

If $\alpha \in \mathrm{AP} \setminus \mathrm{SC}$, then α can be written in normal form

$$\alpha =_{\mathrm{NF}} \varphi_{\alpha_1}(\alpha_2), \qquad \text{with } \alpha_1, \alpha_2 < \alpha.$$

In *Die Normalfunktionen und das Problem der ausgezeichneten Folgen von Ordnungszahlen*, (published 1950 in Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich), Heinz Bachmann in effect used fundamental sequences for ordinals larger than Ω to define a longer hierarchy of Ω-normal functions, and thus gave notations for larger ordinals less than Ω.

He wanted to then assign fundamental sequences $(\nu[\xi])_{\xi<\tau_\nu}$ to these ordinals such that

1. $\tau_\nu \leq \Omega \wedge \big((\tau_\nu \in \mathrm{Lim} \wedge \nu \in \mathrm{Lim}) \vee (\tau_\nu = 1 \wedge \nu = \nu[0]+1)\big)$,
2. for $\nu \neq 0$ we have $\nu = \sup\{\, \nu[\xi]+1 \mid \xi < \tau_\nu \,\}$,
3. if $0 < \xi_1 = \xi_2 + 1 < \tau_\nu$, then either $\nu[\xi_1] \in \mathrm{Lim}$ or $\nu[\xi_1] = \nu[\xi_2]+1$,
4. if $\xi \in \mathrm{Lim} \cap \tau_\nu$, then $\nu[\xi] \in \mathrm{Lim}$, $\tau_{\nu[\xi]} = \xi$, and $\nu[\xi][\eta] = \nu[\eta]$ for $0 \leq \eta < \xi$,
5. if $\nu[\xi] < \zeta \leq \nu[\xi+1]$, then $\nu[\xi] \leq \zeta[0]$.

For each limit $\alpha \leq \Gamma_{\Omega+1}$ define a fundamental sequence $(\alpha[\xi])_{\xi<\tau_\alpha}$:

1. If $\alpha \leq \Omega$, then $\tau_\alpha := \alpha$ and $\alpha[\xi] := \xi$.
2. If $\Omega < \alpha =_{\mathrm{NF}} \beta + \gamma$, then $\tau_\alpha := \tau_\gamma$ and $\alpha[\xi] := \beta + \gamma[\xi]$.
3. If $\Omega < \alpha =_{\mathrm{NF}} \varphi_\beta(\gamma)$ ($\gamma \in \mathrm{Lim}$), then $\tau_\alpha := \tau_\gamma$ and $\alpha[\xi] := \varphi_\beta(\gamma[\xi])$.
4. If $\Omega < \alpha =_{\mathrm{NF}} \varphi_0(\gamma+1) = \omega^{\gamma+1}$, then $\tau_\alpha := \omega$ and $\alpha[\xi] := \omega^\gamma \cdot \xi$.
5. If $\Omega < \alpha =_{\mathrm{NF}} \varphi_{\beta+1}(0)$, then $\tau_\alpha := \omega$ and $\alpha[\xi] := \varphi_\beta^{(\xi)}(0)$.
6. If $\Omega < \alpha =_{\mathrm{NF}} \varphi_{\beta+1}(\gamma+1)$, then $\tau_\alpha := \omega$ and $\alpha[\xi] := \varphi_\beta^{(\xi)}(\varphi_{\beta+1}(\gamma)+1)$.
7. If $\Omega < \alpha =_{\mathrm{NF}} \varphi_\beta(0)$ ($\beta \in \mathrm{Lim}$), then $\tau_\alpha := \tau_\beta$ and $\alpha[\xi] := \varphi_{\beta[\xi]}(0)$.
8. If $\Omega < \alpha =_{\mathrm{NF}} \varphi_\beta(\gamma+1)$ ($\beta \in \mathrm{Lim}$), then $\tau_\alpha := \tau_\beta$ and $\alpha[\xi] := \varphi_{\beta[\xi]}(\varphi_\beta(\gamma)+1)$.
9. If $\alpha = \Gamma_{\Omega+1}$, then $\tau_\alpha := \omega$, $\alpha[0] := \Omega + 1$, $\alpha[\xi+1] := \varphi_{\alpha[\xi]}(0)$, $\alpha[\omega] := \alpha$.

Bachmann used these fundamental sequences to define a more powerful variant of the Veblen hierarchy:

1. $\tilde{\varphi}_0(\xi) := \omega^\xi$.
2. $\tilde{\varphi}_{\alpha+1}(\xi) := \tilde{\varphi}'_\alpha(\xi)$.
3. If $\alpha \in \mathrm{Lim}$ with $\tau_\alpha < \Omega$, then $\tilde{\varphi}_\alpha(\xi) := \mathrm{en}_R(\xi)$ with $R = \bigcap_{\xi < \tau_\alpha} \mathrm{Ran}(\tilde{\varphi}_{\alpha[\xi]})$.
4. If $\alpha \in \mathrm{Lim}$ with $\tau_\alpha = \Omega$, then $\tilde{\varphi}_\alpha(\xi) := \tilde{\varphi}_{\alpha[\xi]}(0)$.

(It is more efficient to directly to the successor in the last case.)

Bachmann then gave refined fundamental sequences for countable ordinals less than $\tilde{\varphi}_{\varphi_\Omega(1)+1}(0)$ (H(1) in Bachmann's article).

By recursion on α we define simultaneously

$$\begin{aligned}\mathrm{B}(\alpha) := {} & \text{the least set containing } \{\,0, \Omega\,\} \\ & \text{and closed under } +\text{, the Veblen function } \lambda\xi\eta.\,\varphi_\xi(\eta), \\ & \text{and the restricted function } \psi \restriction \alpha := \lambda\xi < \alpha.\,\psi(\xi), \\ \psi(\alpha) := {} & \min \Omega \setminus \mathrm{B}(\alpha).\end{aligned}$$

Then we have:

1. Each $\mathrm{B}(\alpha)$ is countable.
2. $\psi(\alpha) < \Omega$ and $\psi(\alpha) \notin \mathrm{B}(\alpha)$.
3. $\psi(\alpha) \in \mathrm{SC}$ (strongly critical ordinals: closed under φ).
4. If $\alpha \leq \beta$, then $\psi(\alpha) \leq \psi(\beta)$ and $\mathrm{B}(\alpha) \subseteq \mathrm{B}(\beta)$.
5. If $\alpha < \beta$ and $\alpha \in \mathrm{B}(\beta)$, then $\psi(\alpha) < \psi(\beta)$.
6. ψ is continuous.

Some ordinals

- ε_0.
- Γ_0 (Feferman-Schütte).
- $\psi(\Omega^\omega)$ (small Veblen ordinal).
- $\psi(\Omega^\Omega)$ (large Veblen ordinal).
- $\psi(\varepsilon_{\Omega+1})$ (Howard ordinal).
- $\psi(\varphi_\Omega(1))$ (Bachmann's H(1)).
- $\psi(\Gamma_{\Omega+1})$ (ordinal of unfolding of ID_1).

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(Fundamental sequences on whiteboard.)

Outline

What is unfolding?

The unfoldings are defined for *schematic systems* in first-order logic.

Schematic systems, S, are formulated with a free predicate variable U accompanied by a rule of formula substitution

(Subst) $$\frac{A(U)}{A(\{\, x \mid B(x)\,\})}$$

Examples of schematic systems

- Non-finitist arithmetic, NFA with

$$U(0) \land (\forall x.\, U(x) \to U(x')) \to \forall x.\, U(x)$$

- Zermelo's set theory with

$$\forall a.\, \exists b.\, \forall x.\, x \in b \leftrightarrow x \in a \land U(x).$$

Operational unfolding

- The universe of the operational unfolding consists of the original sorts of S, embedding into a single sort of operations by means of predicates V_s.
- Each n-ary function symbol f of S determines an total n-ary operation f^* on the corresponding sorts.
- Machinery to define new operations by recursion and explicit definition (the universe forms a PCA, in fact a model of BON).
- Each n-ary predicate symbol R of S determines a predicate R^*.
- The axioms of S are included, relativized to the V_s.
- The logic of $\mathcal{U}_0(\mathrm{S})$ is Beeson's *logic of partial terms*.

We include in $\mathcal{U}_0(\mathrm{S})$ the substitution rule:

$$\frac{A(U)}{A(\{\, x \mid B(x) \,\})}\ (\textsc{Subst})$$

- The language of $\mathcal{U}(\mathrm{S})$ extends the language of $\mathcal{U}_0(\mathrm{S})$ by additional constants for the predicate symbols of S plus $\mathrm{eq}, \mathrm{pr}_U, \mathrm{inv}, \mathrm{neg}, \mathrm{conj}, \mathrm{un}, \mathrm{join}$.

Remark

Our formulation here allows us to focus on the rôle of the join operation. We get the intermediate unfolding by leaving out the join axioms.

- $\Pi(\mathrm{nat}) \wedge \forall x(x \in \mathrm{nat} \leftrightarrow \mathrm{N}(x))$.
- $\Pi(\mathrm{eq}) \wedge \forall x(x \in \mathrm{eq} \leftrightarrow \exists y(x = \langle\!\langle y, y \rangle\!\rangle))$.
- $\Pi(\mathrm{pr}_R) \wedge \forall x(x \in \mathrm{pr}_R \leftrightarrow R^*(x))$.
- $\Pi(a) \to \Pi(\mathrm{inv}(a, f)) \wedge \forall x(x \in \mathrm{inv}(a, f) \leftrightarrow f\,x \in a)$.
- $\Pi(a) \wedge \Pi(b) \to \Pi(\mathrm{conj}(a, b)) \wedge \forall x(x \in \mathrm{conj}(a, b) \leftrightarrow x \in a \wedge x \in b)$.
- $\Pi(a) \to \Pi(\mathrm{neg}\,a) \wedge \forall x(x \in \mathrm{neg}(a) \leftrightarrow \neg(x \in a))$.
- $\Pi(a) \to \Pi(\mathrm{un}\,a) \wedge \forall x(x \in \mathrm{un}(a) \leftrightarrow \forall y(\mathrm{N}(y) \to \langle\!\langle x, y \rangle\!\rangle \in a))$.
- The dependent join axiom:

$$\begin{aligned} &\Pi(a) \wedge (\forall y \in a)\Pi(f\,y) \to \Pi(\mathrm{join}(f, a)) \\ &\quad \wedge \forall x(x \in \mathrm{join}(f, a) \leftrightarrow \exists y, z(x = \langle\!\langle y, z \rangle\!\rangle \wedge y \in a \wedge z \in f(y))). \end{aligned}$$

Full unfolding, substitution

For $\mathcal{U}(\mathrm{S})$ we restrict the substitution rule,

$$\frac{A(U)}{A(\{\, x \mid B(x)\,\})}\ (\textsc{Subst})\ ,$$

by requiring A to be in the language of $\mathcal{U}_0(\mathrm{S})$. This is needed, because the full unfolding language reflects the free predicate U.

Theorem (Feferman and Strahm, 2000)

We have the following proof-theoretic equivalences

- $\mathcal{U}_0(\mathrm{NFA}) \equiv \mathrm{PA}$
- $\mathcal{U}_1(\mathrm{NFA}) \equiv \mathrm{RA}_{<\omega}$
- $\mathcal{U}(\mathrm{NFA}) \equiv \mathrm{RA}_{<\Gamma_0}$

In each case the systems prove the same arithmetical sentences.

Unfolding of ID_1

Feferman proposed to look at $\mathcal{U}(\mathrm{ID}_1)$, as another example of predicative closure.

Formal system for one arithmetical inductive definition, ID_1

The language of ID_1, $\mathcal{L}^1$, is that of first-order arithmetic, PA, (with a free predicative variable U) augmented by predicate symbols $I_{\mathfrak{A}}$ for each arithmetical positive operator form $\mathfrak{A}(X, x)$ that does not contain U.

The universal case is that of Kleene's $\mathcal{O}$ consisting of ordinal notations for recursive ordinals.

Theorem

$\mathcal{U}(\mathrm{ID}_1)$ *has proof-theoretic strength* $\psi(\Gamma_{\Omega+1})$.

Strategy

The strategy for the lower bound proof consists of combining two elements

- A lower bound proof for ID_1 (recall that $|\mathrm{ID}_1| = \psi(\varepsilon_{\Omega+1})$)
- The techniques for reaching a strongly critical ordinal using the predicate unfolding machinery from Feferman and Strahm '00.

There is a simultaneous finitary inductive definitions of terms, α, finite sets $\mathrm{K}(\alpha)$ and an ordering $\alpha < \beta$.

Definition of $\mathrm{SC} \subseteq \mathrm{H} \subseteq \mathrm{ON}$:

- $\langle 0 \rangle \in \mathrm{ON}$ (denoting 0),
- $\langle 1 \rangle \in \mathrm{SC}$ (denoting Ω),
- if $n > 1$, $\alpha_1, \ldots, \alpha_n \in \mathrm{H}$ and $\alpha_1 \geq \cdots \geq \alpha_n$, then $\langle 2, \alpha_1, \ldots \alpha_n \rangle \in \mathrm{ON}$ (denoting $\alpha_1 + \cdots + \alpha_n$),
- $\alpha, \beta \in \mathrm{ON}$, then $\langle 3, \alpha, \beta \rangle \in \mathrm{H}$ (denoting $\bar{\varphi}_\alpha \beta$).
- if $\alpha \in \mathrm{ON}$ and $\mathrm{K}(\alpha) \subseteq \alpha$, then $\langle 4, \alpha \rangle \in \mathrm{SC}$ (denoting $\psi(\alpha)$).

Definition of $\mathrm{K}(\alpha)$:

$$
\begin{aligned}
\mathrm{K}(0) &:= \emptyset, \\
\mathrm{K}(\Omega) &:= \emptyset, \\
\mathrm{K}(\alpha_1 + \cdots + \alpha_n) &:= \mathrm{K}(\alpha_1) \cup \cdots \cup \mathrm{K}(\alpha_n), \\
\mathrm{K}(\bar{\varphi}_\alpha \beta) &:= \mathrm{K}(\alpha) \cup \mathrm{K}(\beta), \\
\mathrm{K}(\psi(\alpha)) &:= \{\, \alpha \,\} \cup \mathrm{K}(\alpha).
\end{aligned}
$$

The notation system, continued

For $\alpha, \beta \in \mathrm{ON}$, put $\alpha < \beta$ if one of the following conditions obtains:

- $\alpha = 0$ and $\beta \neq 0$,
- $\alpha = \alpha_1 + \cdots + \alpha_m$, $\beta = \beta_1 + \cdots + \beta_n$, and either
  - $m \geq n$ and $\exists i \leq n.\, \alpha_i < \beta_i \wedge \forall j < i.\, \alpha_j = \beta_j$, or
  - $m < n$ and $\forall i \leq m.\, \alpha_i = \beta_i$.
- $\alpha = \alpha_1 + \cdots + \alpha_n$, $\beta \in \mathrm{H}$, and $\alpha_1 < \beta$.
- $\alpha \in \mathrm{H}$, $\beta = \beta_1 + \cdots + \beta_n$, and $\alpha \leq \beta_1$.
- $\alpha = \bar{\varphi}_{\alpha_1}\alpha_2$, $\beta = \bar{\varphi}_{\beta_1}\beta_2$ and one of the following obtains
  - $\alpha_1 < \beta_1$ and $\alpha_2 < \beta$.
  - $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$.
  - $\beta_1 > \alpha_1$ and $\beta_2 \leq \alpha$.
- $\alpha = \bar{\varphi}_{\alpha_1}\alpha_2$, $\beta \in \mathrm{SC}$, and $\alpha_1, \alpha_2 < \beta$.
- $\alpha \in \mathrm{SC}$, $\beta = \bar{\varphi}_{\beta_1}\beta_2$, and $\alpha \leq \beta_1$ or $\alpha \leq \beta_2$.
- $\alpha = \psi(\alpha_1)$, $\beta = \psi(\beta_1)$ and $\alpha_1 < \beta_1$.
- $\alpha = \psi(\alpha_1)$ and $\beta = \Omega$.

Lemma

The class $\mathcal{A}cc$ is closed under all the parts of the notation system that are "from below", i.e., 0, $+$, $\bar{\varphi}$.

The lower bound for ID_1, continued

Lemma

- $\mathrm{ID}_1 \vdash \mathrm{TI}_1(\Omega+1, U) \wedge \mathrm{K}(\Omega+1) \subseteq \Omega+1 \wedge \Omega+1 \in \mathrm{M}$.
- *If* $\mathrm{ID}_1 \vdash \mathrm{TI}_1(\alpha, U) \wedge \mathrm{K}(\alpha) \subseteq \alpha \wedge \alpha \in \mathrm{M}$, *then* $\mathrm{ID}_1 \vdash \mathrm{TI}_1(\omega^\alpha, U) \wedge \mathrm{K}(\omega^\alpha) \subseteq \omega^\alpha \wedge \omega^\alpha \in \mathrm{M}$.

Lemma

If $\mathrm{ID}_1 \vdash \mathrm{TI}_1(\alpha, U) \wedge \mathrm{K}(\alpha) \subseteq \alpha \wedge \alpha \in \mathrm{M}$, *then* $\mathrm{ID}_1 \vdash \psi(\alpha) \in \mathcal{A}cc$.

Corollary

For any $\alpha < \psi(\varepsilon_{\Omega+1})$, $\mathrm{ID}_1 \vdash \mathrm{TI}(\alpha, U)$.

Let $A(X, \alpha, x)$ be a formula of ID_1 with at most X, α, x free. We wish to define segments (in terms of the $<_1$-relation) of the A jump hierarchy starting with U, given set-theoretically by the transfinite recursion

$$Y_0 := \{\, x \mid U(x) \,\},$$
$$Y_\alpha := \{\, x \mid A(Y^\alpha, \alpha, x) \,\}$$

where $Y^\alpha := \{\, (\beta, m) \mid \beta <_1 \alpha \wedge m \in Y_\beta \,\}$.
Define a term $\mathrm{hier}_A : (\iota \rightharpoonup \pi_1)$ in $\mathcal{U}(\mathrm{ID}_1)$ by

$$\mathrm{hier}_A := \mathrm{LFP}\Big(\lambda f, \alpha. \{\, \text{if } \alpha = 0 \text{ then } \mathrm{Pr}_U \text{ else } r_A\big(\mathrm{join}(f, (<_1 \alpha)), \alpha\big) \,\}\Big).$$

Note that we really need the *dependent version* of the join operation.

Lemma

If $\mathcal{U}(\mathrm{ID}_1) \vdash \mathrm{TI}_1(\alpha, U)$, *then* $\mathcal{U}(\mathrm{ID}_1) \vdash \forall \beta <_1 \alpha.\, \mathrm{hier}_A(\beta)\!\downarrow$.

By a clever choice of A (following Feferman and Schütte), we obtain:

Lemma

If $\mathcal{U}(\mathrm{ID}_1) \vdash \mathrm{TI}_1(\alpha, U) \wedge \mathrm{K}(\alpha) \subseteq \alpha \wedge \alpha \in \mathrm{M}$, *then*
$\mathcal{U}(\mathrm{ID}_1) \vdash \mathrm{TI}_1(\varphi_\alpha(0), U) \wedge \mathrm{K}(\varphi_\alpha(0)) \subseteq \varphi_\alpha(0) \wedge \varphi_\alpha(0) \in \mathrm{M}$.

Corollary

For any $\alpha < \psi(\Gamma_{\Omega+1})$, $\mathcal{U}(\mathrm{ID}_1) \vdash \mathrm{TI}(\alpha, U)$.

Upper bound strategy

The strategy for the upper bound is:

- Embed $\mathcal{U}(\mathrm{ID}_1)$ in an intermediate system $(\mathrm{ID}_1)^+_{\mathrm{ON}} + (\textsc{Subst})$ (in analogy with Feferman and Strahm '00).
- Interpret $(\mathrm{ID}_1)^+_{\mathrm{ON}} + (\textsc{Subst})$ in infinitary systems for numbers and ordinals based on ID_1.
- Extract the upper bound using cut-elimination and asymmetric interpretation for the infinitary systems.

We introduce a theory $(\mathrm{ID}_1)^+_{\mathrm{ON}} + (\textsc{Subst})$, analogous to the system $\mathrm{PA}^+_\Omega + (\textsc{Subst})$ from Strahm '00.
$(\mathrm{ID}_1)^+_{\mathrm{ON}} + (\textsc{Subst})$ is formulated in the language $\mathcal{L}^1_{\mathrm{ON}}$, which is obtained from the language of ID_1, $\mathcal{L}^1$, by

- adding a new sort for ordinal variables (with $<$ and $=$ relations), and
- an $(n+1)$-ary predicate symbol $P_{\mathfrak{A}}$ for each inductive operator form $\mathfrak{A}(X, \vec{x})$ over ID_1

(that is, $\mathfrak{A}$ is an $\mathcal{L}^1$-formula so it can contain U and both positive and negative occurrences of $I_{\mathcal{O}}$, but of course only positive occurrences of the fresh n-ary predicate variable X).
As a matter of notation we write $P^\alpha_{\mathfrak{A}}(\vec{x})$ instead of $P_{\mathfrak{A}}(\alpha, \vec{x})$, and we put $P^{<\alpha}_{\mathfrak{A}}(\vec{x}) := \exists \beta < \alpha.\, P^\beta_{\mathfrak{A}}(\vec{x})$.

We axiomatize $(\mathrm{ID}_1)^+_{\mathrm{ON}} + (\textsc{Subst})$ by:

- Number-theoretic axioms
- Schematic induction on the natural numbers
- Schematic induction and closure of the arithmetical inductive definition.
- Inductive operator axioms:

$$P^{\sigma}_{\mathfrak{A}}(\vec{x}) \leftrightarrow \mathfrak{A}(P^{<\sigma}_{\mathfrak{A}}, \vec{x}).$$

- Linearity axioms for the ordinals.
- Σ-reflection scheme on the ordinal sort.
- Σ-induction scheme on the ordinal sort.
- Substitution rule: For A an $\mathcal{L}^1$-formula, and $B(x)$ an $\mathcal{L}^1_{\mathrm{ON}}$-formula:

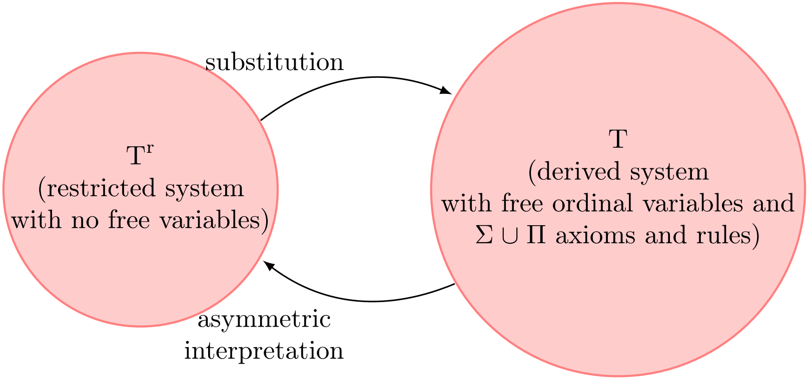
$$\frac{A(U)}{A(\{\, x \mid B(x) \,\})}\ (\textsc{Subst})$$

We embed $\mathcal{U}(\mathrm{ID}_1)$ into $(\mathrm{ID}_1)^+_{\mathrm{ON}} + (\textsc{Subst})$ by

- Interpreting each partial operation by a code for a partial recursive function.
- Writing an inductive operator form that simultaneously defines:
    - A collection Π of (non-unique) codes of predicates of the unfolding.
    - A complimentary pair of relations $\in$ and $\bar{\in}$ that determine the extension for each such code.

  (The dependent join operator causes Π to depend on $\bar{\in}$, for example.)
- Note that the substitution rule of $(\mathrm{ID}_1)^+_{\mathrm{ON}} + (\textsc{Subst})$ interprets the substitution rule of $\mathcal{U}(\mathrm{ID}_1)$.

Asymmetric Interpretation and Substitution



Lemma (Substitution lemma for $\mathrm{T^r}$ into T)

Let $\Gamma(U)$ be a finite set of $\mathcal{L}_\infty^{1,\mathrm{rc}}$-formulas with no occurrences of $P_{\mathfrak{A}}$-literals, and let $B(x)$ be any formula of $\mathcal{L}_\infty^{1,\mathrm{c}}$. Assume $\mathrm{T^r}, \mathcal{H} \left|\frac{\alpha}{\Omega+1}\right. \Gamma(U)$ for some infinite ordinal α. Then we have $\mathrm{T}, \mathcal{H}[\mathrm{par}\, B] \left|\frac{\alpha}{<\omega}\right. \Gamma(\{\, x \mid B(x) \,\})$.

Theorem (Asymmetric interpretation of T into $\mathrm{T^r}$)

Assume Γ is a finite set of $\Sigma \cup \Pi$-formulas of T so that $\mathrm{T}, \mathcal{H} \left|\frac{\alpha}{1}\right. \Gamma$. Let $\beta \geq \Omega$ be a limit ordinal and put $\gamma := \varphi\alpha(\beta + \beta)$. Then for every (β, γ)-instance Λ of Γ with $\mathrm{par}(\Lambda) \cup \{\beta\} \subseteq \mathcal{H}$ we have $\mathrm{T^r}, \mathcal{H} \left|\frac{\gamma}{\gamma}\right. \Lambda$.

Theorem (Reduction of $(\mathrm{ID}_1)^+_{\mathrm{ON}} + (\textsc{Subst})$)

Let C be a formula of $\mathcal{L}^1_{\mathrm{ON}}$, and let A be a closed formula of $\mathcal{L}^1$. Then we have for all natural numbers n, and all acceptable operators $\mathcal{H}$:

1. $(\mathrm{ID}_1)^+_{\mathrm{ON}} + (\textsc{Subst})^{\leq n} \vdash C \quad \rightarrow \quad \mathrm{T}, \mathcal{H} \left|\frac{<\xi_{2n}}{1}\right. C^\star.$
2. $(\mathrm{ID}_1)^+_{\mathrm{ON}} + (\textsc{Subst})^{\leq n} \vdash A \quad \rightarrow \quad \mathrm{T}^{\mathrm{r}}, \mathcal{H} \left|\frac{<\xi_{2n+2}}{\Omega+1}\right. A^\star.$

Theorem (Collapsing Theorem for T^{r})

Let Δ *be a set of* Σ^{Ω}*-sentences of* $\mathcal{L}^{1,\mathrm{r}}_{\infty}$*, and assume* $\mathrm{T}^{\mathrm{r}}, \mathcal{H}_0 \mid\frac{\alpha}{\Omega+1} \Delta$*. Then*

$$\mathrm{T}^{\mathrm{r}}, \mathcal{H}_{\omega^{\Omega+1+\alpha}} \mid\frac{\psi(\omega^{\Omega+1+\alpha})}{\psi(\omega^{\Omega+1+\alpha})} \Delta.$$

Corollary

$|(\mathrm{ID}_1)^{+}_{\mathrm{ON}} + (\textsc{Subst})| \leq \psi(\Gamma_{\Omega+1})$.

Open problems

- Determine the strength of $\mathcal{U}_1(\mathrm{ID}_1)$ (no join).
- Determine the strength of $\mathcal{U}_2(\mathrm{ID}_1)$ (restricted join).
- Give a system of strength $\psi(\varphi_\Omega(1))$.

Questions or Comments?