The quaternionic Hopf fibration in HoTT via a modified Cayley-Dickson construction

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MPIM Bonn, February 12, 2016

Definition An *algebra* is a vector space A (over $\mathbb{R}$) equipped with a bilinear map $m : A \times A \to A$ and an element $1 \in A$ such that $1a = a = a1$ (using juxtaposition to denote application of m).
A $*$-algebra is an algebra A equipped with a conjugation map, a linear map $A \to A$ written $a \mapsto a^*$ with

$$a^{**} = a \qquad (ab)^* = b^* a^*.$$

Let A be a $*$-algebra, and define a $*$-algebra structure on $A' := A \oplus A$ by:

$$\begin{aligned} 1 &:= (1, 0) \\ (a, b)(c, d) &:= (ac - db^*, a^* d + cb) \\ (a, b)^* &:= (a^*, -b) \end{aligned}$$

Definition A is *real* iff $a^* = a$ for all a.
A is *nicely normed* iff $a + a^* \in \mathbb{R}$ and $aa^* = a^*a > 0$ for all $a \neq 0$.

Properties of the Cayley-Dickson construction:

- A' is never real.
- A is real (and thus commutative) iff A' is commutative.
- A is commutative and associative iff A' is associative.
- A is associative and nicely normed iff A' is alternative and nicely normed.
- A is nicely normed iff A' is nicely normed.

Definitions A is *power-associative* iff the subalgebra generated by any one element is associative.
A is *alternative* iff the subalgebra generated by any two elements is associative.
A is a *division algebra* iff $ab = 0$ implies $a = 0$ or $b = 0$.
A is a *normed division algebra* if A is equipped with a norm $\|-\|$ such that $\|ab\| = \|a\|\|b\|$

Proposition If A is nicely normed and alternative, then A is a normed division algebra under the norm

$$\|a\|^2 = aa^*$$

Starting from the real numbers, we get in succession:

- $\mathbb{R}$, a real commutative associative nicely normed $*$-algebra;
- $\mathbb{C}$, a commutative associative nicely normed $*$-algebra;
- $\mathbb{H}$, an associative nicely normed $*$-algebra;
- $\mathbb{O}$, an alternative nicely normed $*$-algebra.
- the *sedenions*, a 16-dimensional nicely normed $*$-algebra.

Theorem (Hurwitz 1898) $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are the only normed division algebras.
Theorem (Zorn 1930) $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are the only alternative division algebras.
Theorem (Kervaire, Bott-Milnor 1958) All division algebras have dimension $1, 2, 4$ or 8.

The Hopf fibrations

From the classical normed division algebras, we get H-space structures on S^0, S^1, S^3 and S^7. (An H-space is a unital magma in the homotopy category.)

And from these we in turn get fibrations

- $S^0 \hookrightarrow S^0 * S^0 = S^1 \to S^1$ (the real Hopf fibration/the Möbius bundle).
- $S^1 \hookrightarrow S^1 * S^1 = S^3 \to S^2$ (the (complex) Hopf fibration).
- $S^3 \hookrightarrow S^3 * S^3 = S^7 \to S^4$ (the quaternionic Hopf fibration).
- $S^7 \hookrightarrow S^7 * S^7 = S^{15} \to S^8$ (the octonionic Hopf fibration).

Of these, we have the first two in HoTT (and in several formalizations).
We would like to have the other two as well!

Outline

Which HoTT? MLTT with a predicative univalent universe, indexed inductive types, and two classes of built-in HITs: n-truncation and homotopy quotients (equivalently, homotopy coequalizers).

This is the basis for the HoTT mode in the Lean proof assistant in which the present work has been formalized.

The homotopy quotient of $A : \mathrm{Type}$ and $R : A \to A \to \mathrm{Type}$ has

- Point constructor: $\mathrm{class\_of} : A \to \mathrm{quotient}\, R$,
- Path constructor: $\mathrm{eq\_of\_rel} : \Pi a, a' : A, R\, a\, a' \to \mathrm{class\_of}\, a = \mathrm{class\_of}\, a'$.

From this we build pushouts, suspensions, colimits, cell complexes, ... (and in fact also -1-truncation, due to Floris van Doorn).

First attempt to port Cayley-Dickson construction

Definition A *Cayley-Dickson spheroid* consists of an H-space S (we write 1 for the base point, and concatenation denotes multiplication) with additional operations

(conjugation)	$x \mapsto x^*$	$: S \to S$
(negation)	$x \mapsto -x$	$: S \to S$

satisfying the further laws

$$\begin{aligned} 1^* &= 1 & (-x)^* &= -x^* \\ -(-x) &= x = x^{**} & x(-y) &= -xy \\ (xy)^* &= y^* x^* & x^* x &= 1. \end{aligned}$$

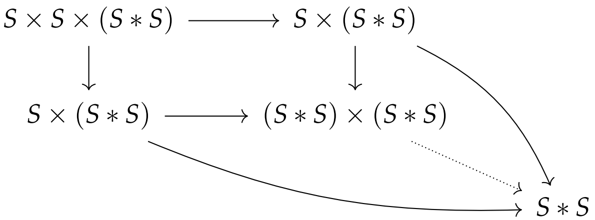
NB It follows that $xx^* = 1$ and $(-x)y = -xy$ as well.

Hope

We would like to then give an H-space structure on $S * S$ under the assumption that S is associative.

We wish to define the multiplication xy for $x, y : S * S$ by induction on x (first) and y (second).

To do the induction on x we must define elements $(\mathrm{inl}\ a)y$, $(\mathrm{inl}\ b)y$ and dependent paths $(\mathrm{glue}\ ab)_* y : (\mathrm{inl}\ a)y =_{\mathrm{glue}\ ab} (\mathrm{inr}\ b)y$ for $a, b : S$. This is of course the same as giving the two bent arrows such that the outer square commutes in this diagram:



In each case we do an induction on y, giving the following point constructor problems, which we solve by appealing to the classical Cayley-Dickson construction:

$$\begin{array}{ll} (\mathrm{inl}\ a)(\mathrm{inl}\ c) := \mathrm{inl}(ac) & (\mathrm{inl}\ a)(\mathrm{inr}\ d) := \mathrm{inr}(a^* d) \\ (\mathrm{inr}\ b)(\mathrm{inl}\ c) := \mathrm{inr}(cb) & (\mathrm{inr}\ b)(\mathrm{inr}\ d) := \mathrm{inl}(-db^*) \end{array}$$

cf.:

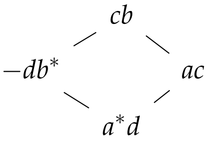
$$(a,b)(c,d) := (ac - db^*, a^* d + cb)$$

Path constructor problems

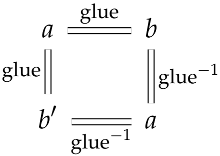
We must define four dependent paths corresponding to the interaction of a point constructor with a path constructor, and these we all fill with glue (or its inverse).

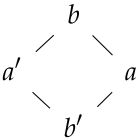
Finally, we must give a pathover in an identity type family, which we can think of as the problem of filling the square on the left, also depicted on the right as a *diamond*:

$$\begin{array}{ccc} \mathrm{inl}(ac) & \overset{\mathrm{glue}}{=\!=\!=} & \mathrm{inr}(cb) \\ \mathrm{glue} \Vert & & \Vert \mathrm{glue}^{-1} \\ \mathrm{inr}(a^*d) & \underset{\mathrm{glue}^{-1}}{=\!=\!=} & \mathrm{inl}(-db^*) \end{array}$$



These diamond shapes will play an important role in the construction. We can define these diamond types as certain square types sitting in any join, $A * B$, for any $a, a' : A$ and $b, b' : B$:





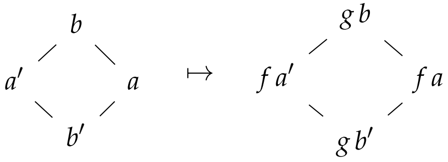
The geometric intuition behind the shape is that we picture the join $A * B$ as A lying on a horizontal line, B on a vertical line, and glue-paths connecting every point in A to every point in B.

Problem: Given a diamond corresponding to $a, a' : A$ and $b, b' : B$, and given either a path $p : a =_A a'$ or a path $q : b =_B b'$, we are to fill the diamond (i.e., inhabit the square type).

Construction: We perform path induction on the given path (p or q). Then we reduce to a standard open box filling problem.

Solving diamond problems, continued

Problem: Given types types A_1, A_2, B_1, B_2 and functions $f : A_1 \to A_2$ and $g : B_1$ to B_2, and given a solution to the diamond problem in $A_1 * B_1$ given by $a, a' : A_1$, $b, b' : B_1$, to obtain a solution to the diamond problem in $A_2 * B_2$ given by $f\,a, f\,a' : A_2$, $g\,b, g\,b' : B_2$:



Construction: This is a special case of functoriality of square types, using the induced function $f * g : A_1 * B_1 \to A_2 * B_2$.

Returning to our original diamond problem

Recall our problem: to fill the diamond:

$$\begin{array}{ccc} & cb & \\ -db^* & & ac \\ & a^*d & \end{array}$$

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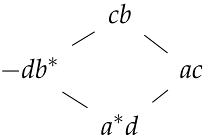
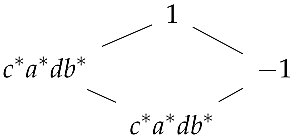


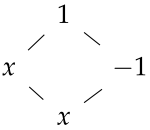
If S is associative, we have maps $f, g : S \to S$:

$$f(x) := -acx, \qquad g(y) := cyb$$

Then we have $f(-1) = ac$, $f(c^*a^*db^*) = -db^*$, $g(1) = cb$, and $g(c^*a^*db^*) = a^*d$. Thus, it suffices to solve the diamond problem,



or simply,



with $x = c^*a^*db^*$.

Outline

New idea Axiomatize the unit imaginaries in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.

Note that both conjugation and negation on the unit sphere are determined by the negation acting on the unit imaginaries. In fact, we can make the following general constructions:
Suppose A is a type with a negation. Then we can define a conjugation and a negation on the suspension of A, ΣA:

$$
\begin{aligned}
\mathrm{N}^* &:= \mathrm{N} & -\mathrm{N} &:= \mathrm{S} \\
\mathrm{S}^* &:= \mathrm{S} & -\mathrm{S} &:= \mathrm{N} \\
(\mathrm{merid}\ a)^* &:= \mathrm{merid}(-a) & -\mathrm{merid}\ a &:= \mathrm{merid}(-a)^{-1}
\end{aligned}
$$

We give ΣA the basepoint N, which we also write as 1. If the negation on A is involutive, then so is the conjugation and negation on ΣA.

Definition: A *Cayley-Dickson imaginaroid* consists of a type A with an involutive negation, together with a binary multiplication operation on ΣA, such that ΣA becomes an H-space, and satisfying the further laws

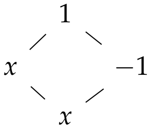
$$\begin{aligned} x(-y) &= -xy \\ xx^* &= 1 \\ (xy)^* &= y^* x^* \end{aligned}$$

for $x, y : \Sigma A$.

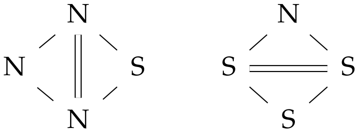
NB If A is a Cayley-Dickson imaginaroid, then ΣA becomes a Cayley-Dickson spheroid.

Problem: If A is an associative imaginaroid, give $\Sigma A * \Sigma A$ an H-space structure.

Construction: By the previous remarks, it suffices to solve the diamond problem:

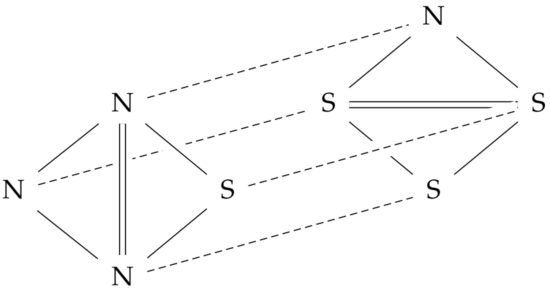


Now perform induction on $x : \Sigma A$. We get problems for the poles:



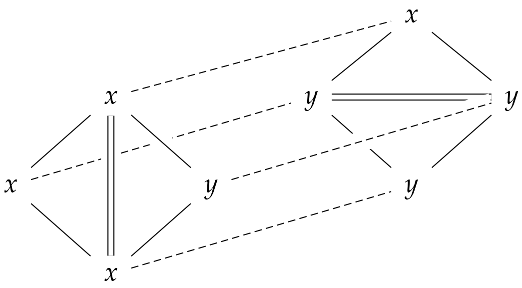
The resulting pathover

These solutions must now be connected by filling, for every $a : A$, the following hollow cube connecting the diamonds:



Here, the two dashed paths $\mathrm{N} = \mathrm{N}$ and $\mathrm{S} = \mathrm{S}$ are identities, while the other two are each the meridian, $\mathrm{merid}\ a : \mathrm{N} = \mathrm{S}$.

Generalizing a bit, we see that we can fill any cube in a symmetric join, $B * B$, with $p : x =_B y$, of this form:



Indeed, this follows by path induction on p.

Quod Erat Faciendum.

Outline

Getting the quaternionic Hopf fibration

Problem: Give an H-space structure on S^3

Construction: Since $S^3 = \Sigma S^0 * \Sigma S^0$, it suffices to give an imaginaroid structure on S^0. This is not too difficult.

- S^3 is not just an H-space, but a group, that is, find a type BS^3 such that $S^3 = \Omega BS^3$ (a model for BS^3 is $\mathbb{H}P^\infty$, just like $BS^0 = \mathbb{R}P^\infty$ and $BS^1 = \mathbb{C}P^\infty$.

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Where to go from here

- S^3 is not just an H-space, but a group, that is, find a type BS^3 such that $S^3 = \Omega BS^3$ (a model for BS^3 is $\mathbb{H}P^\infty$, just like $BS^0 = \mathbb{R}P^\infty$ and $BS^1 = \mathbb{C}P^\infty$.
- With a bit more work, this would give the H-space structure on S^7 and hence the octonionic Hopf fibration.
- Alternatively, add coherence conditions to imaginaroids A sufficient to conclude that $A * \Sigma A$ is a not-necessarily coherent imaginaroid.
- Classically, the Cayley-Dickson construction also gives the exceptional Lie group G_2 as the zero-divisors in the sedenions. It seems an entirely different approach is needed to define G_2 (and its classifying type) in HoTT.