

Notes on higher groups and projective spaces

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1 Truncatedness and connectedness

Recall that we define $\text{istrunc}_n A$ by recursion on $n : \mathbb{N}_{-2}$:

$$\begin{aligned}\text{istrunc}_{-2} A &:= \text{iscontr } A = (a : A) \times ((x : A) \rightarrow (a = x)) \\ \text{istrunc}_{n+1} A &:= (x \ y : A) \rightarrow \text{istrunc}_n(x = y)\end{aligned}$$

Then we define $\text{isconn}_n A := \text{iscontr} \|\!|A\|\!|_n$.

The type of pointed types is $\text{Type}_{\text{pt}} = (A : \text{Type}) \times (\text{pt} : A)$. The type of n -truncated and n -connected types are $\text{Type}^{\leq n} := (A : \text{Type}) \times \text{istrunc}_n A$, $\text{Type}^{> n} := (A : \text{Type}) \times \text{isconn}_n A$.

Theorem 1. *For $A : \text{Type}_{\text{pt}}$, we have for $n \geq -1$:*

- *A is n -truncated only if $\pi_k A = 0$ for $k > n$ (and “if” holds when A is hypercomplete).*
- *A is n -connected if and only if $\pi_k A = 0$ for $k \leq n$.*

For $A : \text{Type}_{\text{pt}}$ we define the n -connected cover of A to be $A\langle n \rangle := \text{fib}(A \rightarrow \|\!|A\|\!|_n)$.

Theorem 2. *For $A : \text{Type}_{\text{pt}}$, $n \geq -2$ and $k \geq 0$ we have*

$$\|\!|\Omega^k A\|\!|_n = \Omega^k \|\!|A\|\!|_{n+k}.$$

2 Higher groups

Recall that types in HoTT may be viewed as ∞ -groupoids: elements are objects, paths are morphisms, higher paths are higher morphisms, etc.

It follows that *pointed connected* types A may be viewed as higher groups, with *carrier* $\Omega A = (\text{pt} = \text{pt})$.

Writing G for the carrier, it's common to write BG for the pointed connected type such that $G = \Omega BG$. Let us write

$$\text{Grp} := (G : \text{Type}) \times (BG : \text{Type}_{\text{pt}}^{>0}) \times (G = \Omega BG) = \text{Type}_{\text{pt}}^{>0}$$

for the type of higher groups. *N.B.* For $G : \text{Grp}$ we also have $G : \text{Type}$ using the first projection as a coercion. Using the last definition, this is the loop space map, and not the usual coercion!

$k \setminus n$	0	1	2	\dots	∞
0	pointed set	pointed groupoid	pointed 2-groupoid	\dots	pointed ∞ -groupoid
1	group	2-group	3-group	\dots	∞ -group
2	abelian group	braided 2-group	braided 3-group	\dots	braided ∞ -group
3	— ” —	symmetric 2-group	symplectic 3-group	\dots	symplectic ∞ -group
4	— ” —	— ” —	symmetric 3-group	\dots	?? ∞ -group
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
ω	— ” —	— ” —	— ” —	\dots	connective spectrum

Table 1: Periodic table of k -tuply groupal n -groupoids.

We recover¹ the ordinary set-level groups by requiring that G is a 0-type, or equivalently, that BG is a 1-type. This leads us to introduce

$$n\text{Grp} := (G : \text{Type}^{\leq n}) \times (BG : \text{Type}_{\text{pt}}^{\geq 0}) \times (G = \Omega BG) = \text{Type}_{\text{pt}}^{\geq 0, \leq n+1}$$

for the type of *groupal* (group-like) n -groupoids, also known as $(n+1)$ -groups. For $G : 0\text{Grp}$ a set-level group, we have $BG = K(G, 1)$.

Of course, double loop spaces are even better behaved than mere loop spaces (e.g., they are commutative up to homotopy). Say a type G is *k-tuply groupal* if we have a k -fold delooping, $B^k G : \text{Type}_{\text{pt}}^{\geq k}$, such that $G = \Omega^k B^k G$.

Mixing the two directions, let us introduce the type

$$(n, k)\text{Grp} := (G : \text{Type}^{\leq n}) \times (B^k G : \text{Type}_{\text{pt}}^{\geq k}) \times (G = \Omega^k B^k G) = \text{Type}_{\text{pt}}^{\geq k, \leq n+k}$$

for the type of *k-tuply groupal n-groupoids*.²

We can also allow k to be infinite, $k = \omega$, but in this case we can't cancel out the G and we must record all the intermediate delooping steps:

$$(n, \omega)\text{Grp} := (B^{-1}G : (k : \mathbb{N}) \rightarrow \text{Type}_{\text{pt}}^{\geq k, \leq n+k}) \times ((k : \mathbb{N}) \rightarrow B^k G = \Omega B^{k+1} G)$$

When $n = \infty$, this is the type of stably groupal ∞ -groups, also known as *connective spectra*. If we also relax the connectivity requirement, we get the type of all spectra, and we can think of a spectrum as a kind of ∞ -groupoid with k -morphisms for all $k \in \mathbb{Z}$.

3 Constructions with higher groups

Here are some constructions with higher groups (giving the action on the carriers as well for clarity, but omitting the third components for readability):

decategorification $\text{Decat} : (n, k)\text{Grp} \rightarrow (n-1, k)\text{Grp}$
 $\langle G, B^k G \rangle \mapsto \langle \|G\|_{n-1}, \|B^k G\|_{n+k-1} \rangle$

¹this requires some honest toil!

²this is called $n\text{Type}_k$ in [1], but here we give equal billing to n and k .

discrete categorification $\text{Disc} : (n, k)\text{Grp} \rightarrow (n + 1, k)\text{Grp}$
 $\langle G, B^k G \rangle \mapsto \langle G, B^k G \rangle$, and $\text{Disc} \dashv \text{Decat}$ with $\text{Decat} \circ \text{Disc} = \text{id}$

looping $\Omega : (n, k)\text{Grp} \rightarrow (n - 1, k + 1)\text{Grp}$
 $\langle G, B^k G \rangle \mapsto \langle \Omega G, B^k G \langle k \rangle \rangle$

delooping $B : (n, k)\text{Grp} \rightarrow (n + 1, k - 1)\text{Grp}$
 $\langle G, B^k G \rangle \mapsto \langle \Omega^{k-1} B^k G, B^k G \rangle$, and $B \dashv \Omega$ with $\Omega \circ B = \text{id}$

forgetting $F : (n, k)\text{Grp} \rightarrow (n, k - 1)\text{Grp}$
 $\langle G, B^k G \rangle \mapsto \langle G, \Omega B^k G \rangle$

stabilization $S : (n, k)\text{Grp} \rightarrow (n, k + 1)\text{Grp}$
 $\langle G, B^k G \rangle \mapsto \langle SG, \|\Sigma B^k G\|_{n+k+1} \rangle$,
where $SG = \|\Omega^{k+1} \Sigma B^k G\|_n$ and $S \dashv F$.

Theorem 3 (Freudenthal). *If $A : \text{Type}_{\text{pt}}^{>n}$ with $n \geq 0$, then the map $A \rightarrow \Omega \Sigma A$ is $2n$ -connected.*

Corollary 1 (Stabilization). *If $k \geq n + 2$, then $S : (n, k)\text{Grp} \rightarrow (n, k + 1)\text{Grp}$ is an equivalence, and any $G : (n, k)\text{Grp}$ is an infinite loop space.*

For example, for $G : (0, 2)\text{Grp}$ an abelian group, we have $B^n G = K(G, n)$, an Eilenberg-MacLane space.

The adjunction $S \dashv F$ implies that the free group on a pointed set X is $\Omega \|\Sigma X\|_1 = \pi_1(\Sigma X)$. If X has decidable equality, ΣX is already 1-truncated. It is an open problem whether this is true in general.

Also, the abelianization of a set-level group $G : 0\text{Grp}$ is $\pi_2(\Sigma BG)$. If $G : (n, k)\text{Grp}$ is in the stable range ($k \geq n + 2$), then $SFG = G$.

4 Homomorphisms and automorphisms

For $G, H : (n, k)\text{Grp}$, define $\text{hom}_{(n,k)}(G, H) := (B^k G \rightarrow_{\text{pt}} B^k H)$. For spectra we need pointed maps between all the deloopings and pointed homotopies showing they cohere.

Note that if $h, k : G \rightarrow H$ are homomorphisms between set-level groups, then h and k are *conjugate* if $Bh, Bk : BG \rightarrow_{\text{pt}} BH$ are *freely* homotopic (i.e., equal as maps $BG \rightarrow BH$).

Also observe that $\pi_j(B^k G \rightarrow_{\text{pt}} B^k H) = \|\Sigma^j B^k G \rightarrow_{\text{pt}} B^k H\|_0 = 0$ for $j + k - 1 \geq n + k$, that is, for $j > n$, so $\text{hom}_{(n,k)}(G, H)$ is actually n -truncated.

If $k \geq n + 2$ (so we're in the stable range), then $\text{hom}_{(n,k)}(G, H)$ becomes a stably groupal n -groupoid. This generalizes the fact that the homomorphisms between abelian groups form an abelian group.

Given *any* type of objects U , any $a : U$ has an *automorphism group* $\text{aut } a = (a = a)$ with $B \text{aut } a = \text{im}(a : 1 \rightarrow U) = (x : U) \times \|a = x\|_{-1}$ (the connected component of U at a). Clearly, if U is $n + 1$ -truncated, then so is $B \text{aut } a$ and so $\text{aut } a$ is n -truncated.

Now, the automorphism group $\text{aut } G$ of a $G : (n, k)\text{Grp}$ is in $(n, 1)\text{Grp}$. But we can also forget the basepoint and consider the automorphism group $\text{aut}^c G$ of $B^k G : \text{Type}^{\geq k, \leq n+k}$. This now allows for (higher) conjugations. We define the *generalized center* of G to be $ZG := \Omega^k \text{aut}^c G : (n, k + 1)\text{Grp}$ (generalizing the center of a set-level group).

5 Group actions

In this section we consider a fixed group $G : \text{Grp}$ with delooping BG . An *action* of G on some object of type U is simply a function $X : BG \rightarrow U$. The object of the action is $X(\text{pt}) : U$, and it can be convenient to consider evaluation at $\text{pt} : BG$ to be a coercion from actions of type U to U . To equip $a : U$ with a G -action is to give an action $X : BG \rightarrow U$ with $X(\text{pt}) = a$. The *trivial action* is the constant function at a . Clearly, an action of G on $a : U$ is the same as a homomorphism $G \rightarrow \text{aut } a$.

If U is a universe of types, then we have actions on types. If X is an action on types, then we can form the:

invariants $X^{hG} := (x : BG) \rightarrow X(x)$, also known as the *homotopy fixed points*

coinvariants $X_{hG} := (x : BG) \times X(x)$, also known as the *homotopy quotient* $X // G$.

Every group G carries two canonical actions on itself:

the right action $G : BG \rightarrow \text{Type}$, $G(x) = (\text{pt} = x)$

the adjoint action $G^{\text{ad}} : BG \rightarrow \text{Type}$, $G^{\text{ad}}(x) = (x = x)$ (by conjugation).

We have $G // G = 1$ and $G^{\text{ad}} // G = LBG = (S^1 \rightarrow BG)$, the free loop space of BG . By definition, BG classifies *principal G -bundles*: pullbacks of the right action of G .

6 Projective spaces

Consider the sequence of actions $GM^n : BG \rightarrow \text{Type}$ of G given by

$$\begin{aligned} GM^{-1}(x) &:= 0 \\ GM^{n+1}(x) &:= (\text{pt} = x) * GM^n(x) = G(x) * GM^n(x) \end{aligned}$$

i.e., the iterated joins of the right action with itself (M is for Milnor). The types $GM^n(\text{pt})$ are at least $(n+1)(k+2)-2$ -connected if G is k -connected. In fact, the colimit $GM^\infty(\text{pt}) = \varinjlim GM^n(\text{pt})$ is contractible, so $GM^\infty // G = 1 // G = BG$. We define the *projective spaces* for G to be $GP^n := GM^n // G$. Thus, $GP^{-1} = 0$, $GP^0 = 1$, $GP^1 = \Sigma G$, etc. (in general, GP^{n+1} is the mapping cone on the inclusion $GM^n(\text{pt}) \rightarrow GP^n$).

The *real* and *complex projective spaces* are $\mathbb{R}P^n := O(1)P^n$ and $\mathbb{C}P^n := U(1)P^n$, where $O(1)$ is the 2-element group, and $U(1)$ is the circle group. Note that $O(1)M^n = S^n$ and $U(1)M^n = S^{2n+1}$.

Given any group G , the *Hopf construction* gives a principal G -bundle $G \hookrightarrow G * G \rightarrow \Sigma G = GP^1$, classified by the projection $H : GP^1 \rightarrow BG$, which corresponds under the *clutching construction* to the identity on G . The clutching construction is simply taking the adjoint of a map $A \rightarrow \Omega BG$ to give a map $\Sigma A \rightarrow_{\text{pt}} BG$.

References

- [1] John C. Baez and James Dolan. ‘‘Categorification’’. In: *Higher category theory (Evanston, IL, 1997)*. Vol. 230. Contemp. Math. Amer. Math. Soc., Providence, RI, 1998, pp. 1–36. DOI: 10.1090/conm/230/03336.