Functional interpretation and inductive definitions

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Introduction

In this talk we will discuss a new way to extract constructive content from the theory ID_1, based on the 2009 JSL paper by Avigad and Towsner: Functional interpretation and inductive definitions.

Inductive Definitions

A reminder: The classical theory ID_1 is Peano Arithmetic extended with predicates I_ψ intended to denote least fixpoints of positive arithmetical operators $\psi(x, P)$. Here P is a fresh predicate variable occuring only positively in ψ (which we take to be in negation normal form), and x is the only free variable in ψ.

ID_1 then includes the axioms:

$\forall x\, (\psi(x, I_\psi) \to x \in I_\psi)$, and

$\forall x\, (\psi(x, \lambda y.\, \theta(y)) \to \theta(x)) \to \forall x \in I_\psi\, \theta(x)$ for each formula $\theta(y)$.

These axioms mean that I_ψ is smallest set closed wrt ψ among sets definable in the language.

Inductive Definitions, continued

ID_1 has the same strength as the subsystem Π^1_1-CA^- of second order arithmetic, which is also the strength of $\mathsf{KP}\omega$; admissible set theory with infinity.

A key goal is to extract computational information from proofs of Π^0_2-sentences in ID_1. Ideally, we do this by a translation from ID_1 to a constructive theory of functionals, allowing us to read off a witnessing function.

Previously, this was done by first reducing ID_1 to $\mathsf{ID}_1^{\mathsf{i,sp}}$ using either

- ordinal analysis (Buchholz, Pohlers); *or*
- a forcing intrepretation (Buchholz).

The constructive theory $\mathsf{ID}_1^{\mathsf{i,sp}}$ can then be analyzed by realizability or a Dialectica-style functional interpretation.

The paper by Avigad and Towsner provides a new route.

History of functional interpretation and ID_1

Gödel, 1958: Functional interpretation of HA (Dialectica).

Feferman, 1968: Ordinal analysis bounds of ID_1 through functional interpretation.

Howard, 1972: Functional interpretation of $\mathsf{ID}_1^{\mathsf{i,sp}}$.

Diller and Nahm, 1974: Variant of Dialectica interpretation not requiring decidability of ground fomulas.

Buchholz, Feferman, Pohlers and Sieg, 1975–1981: Ordinal analysis of ID_ν; equivalence to $\mathsf{ID}_\nu^{\mathsf{i,acc}}$.

Avigad and Feferman, *Handbook*, 1998: posed the problem of reducing ID_1 to a constructively acceptable system via functional interpretation.

Burr, 1997: Functional interpretation of CZF and KPω.

Avigad and Towsner, 2009: Functional interpretation for extracting ordinal bounds.

Functionals over ω

We start with Gödel's theory T of functionals of finite type over ω. The finite types contain ω, and if σ and τ are finite types, then so are $\sigma \to \tau$ and $\sigma \times \tau$.
The terms of T are the primitive recursive functionals of finite type; obtained from zero, successor on ω, using abstraction, application, pairing, projection, and primitive recursion on ω at all types.
The theory T deduces quantifier-free formulas starting from defining equations of functionals using the rule of quantifier free induction:

$$\frac{\varphi(0) \qquad \varphi(x) \to \varphi(Sx)}{\varphi(t)}$$

Functionals over ω and Ω

The theory T_Ω extends Gödel's T by adding a new base type Ω, intended to denote well-founded, ω-branching trees. We add a constant e, denoting the empty tree, and function symbols

$$\sup\colon (\omega \to \Omega) \to \Omega \qquad \text{and} \qquad (-)[-]\colon \Omega \times \omega \to \Omega$$

We have primitive recursion over ω at all types, as well as the following schema of primitive recursion over Ω:

$$F(\alpha) = \begin{cases} a, & \text{for } \alpha = e; \\ G(\lambda n\, F(\alpha[n])), & \text{otherwise.} \end{cases}$$

Here $a\colon \sigma$, $G\colon (\omega \to \sigma) \to \sigma$, and $F\colon \Omega \to \sigma$, so the recursor has type

$$\sigma \to ((\omega \to \sigma) \to \sigma) \to \Omega \to \sigma.$$

Note on the recursor for Ω

For those familiar with non-algebraic inductive types, it might seem that the recursor is too weak. In a dependently-typed setting, the recursor would have type

$$\begin{aligned}
&(P\colon \Omega \to \mathsf{Type}) \to \\
&P\, e \to \\
&((h\colon \omega \to \Omega) \to ((n\colon \omega) \to P\,(h\, n)) \to P\,(\sup h)) \to \\
&(\alpha\colon \Omega) \to P\, \alpha
\end{aligned}$$

When the goal type is a constant, σ, we get

$$\sigma \to ((\omega \to \Omega) \to (\omega \to \sigma) \to \sigma) \to \Omega \to \sigma$$

However, this recursor is definable from the one in T_Ω by pairing up the Ω-argument in the result type, and then projecting the answer in the end.

Overview of the constructive systems

The theory T_Ω deduces quantifier-free formulas using the induction scheme of T. A Trick due to Kreisel shows that T_Ω derives a rule of quantifier free tree induction:

$$\frac{\varphi(e,x) \qquad \alpha \neq e \wedge \varphi(\alpha[g(\alpha,x)], h(\alpha,x)) \to \varphi(\alpha,x)}{\varphi(s,t)}$$

The classical theory QT_Ω extends T_Ω by allowing quantifiers at all types and a full transfinite induction scheme:

$$\varphi(e) \wedge \forall\alpha\,(\alpha \neq e \wedge \forall n\,\varphi(\alpha[n]) \to \varphi(\alpha)) \to \forall\alpha\,\varphi(\alpha)$$

$\mathsf{QT}^{\mathsf{i}}_\Omega$ is QT_Ω over intuitionistic logic.
It is possible to also add the scheme of ω-bounding:

$$\forall x\,\exists\alpha\,\psi(x,\alpha) \to \exists\beta\,\forall x\,\exists n\,\psi(x,\beta[n])$$

Overview of the constructive systems, II

Theorem (2.4)

The following theories prove the same Π^0_2-sentences:

(i) $\mathsf{ID}_1^{\mathsf{i,sp}}$

(ii) $\mathsf{ID}_1^{\mathsf{i,acc}}$

(iii) $\mathsf{QT}^{\mathsf{i}}_{\Omega} + (\omega\text{-}bounding)$

(iv) $\mathsf{QT}^{\mathsf{i}}_{\Omega}$

(v) T_{Ω}

Overview of the paper

Avigad and Towsner interprets ID_1 in three steps:

$$\begin{aligned} \mathsf{ID}_1 &\rightsquigarrow \mathsf{OID}_1 && \text{(embedding, §3)} \\ &\rightsquigarrow \mathsf{Q}_0\mathsf{T}_\Omega + (I) && \text{(functional interpretation, §4)} \\ &\rightsquigarrow \mathsf{QT}^{\mathsf{i}}_\Omega && \text{(cut elimination, §5)} \end{aligned}$$

Theorem (2.5)

Every Π^0_2-sentence provable in ID_1 is provable in $\mathsf{QT}^{\mathsf{i}}_\Omega$.

Theorem (2.6)

If $\mathsf{ID}_1 \vdash \forall \bar{x}\, \exists \bar{y}\, R(\bar{x}, \bar{y})$, then there is a sequence of functions symbols $\bar{f}$ so that $T_\Omega \vdash R(\bar{x}, \bar{f}\bar{x})$.

Staged inductive definitions

The first step is to make the stages of inductive definitions explicit. Fix an instance of ID_1 with inductive predicate I defined by the positive arithmetical operator $\psi(x, P)$. The corresponding instance of the theory OID_1 is two-sorted over types ω and Ω. It includes primitive recursive function symbols over ω, a constant $e \in \Omega$, and a function symbol

$$\Omega \ni \alpha, \omega \ni n \mapsto \alpha[n] \in \Omega.$$

Here $\alpha[n]$ is intended to denote the n'th subtree of α, or e if $\alpha = e$. The language includes equality over ω, but not over Ω. There's just a unary predicate $\lambda\alpha.\, \alpha = e$. Finally, there's a binary predicate $x \in I_\alpha$. We write $x \in I_{\prec\alpha}$ for $\exists n\, (x \in I_{\alpha[n]})$.

Staged inductive definitions, axioms

The axioms of OID_1 are:

(1) defining axioms for the primitive recursive functions

(2) induction on ω

(3) transfinite induction on Ω

(4) $\alpha = e \to \alpha[n] = e$

(5) the schema of ω-*bounding*:

$$\forall x \, \exists \alpha \, \varphi(x, \alpha) \to \exists \beta \, \forall x \, \exists n \, \varphi(x, \beta[n]),$$

where φ has no quantifiers over Ω.

(6) $\forall x \, (x \notin I_e)$

(7) $\forall \alpha \, (\alpha \neq e \to \forall x \, (x \in I_\alpha \leftrightarrow \psi(x, I_{\prec \alpha})))$

Embedding ID_1 in OID_1

For any formula φ of ID_1, let $\hat{\varphi}$ be the formula obtained by replacing each occurance of $t \in I$ with $\exists\alpha\,(t \in I_\alpha)$.

Theorem (3.1)

If ID_1 *proves* φ*, then* OID_1 *proves* $\hat{\varphi}$*.*

This is deduced from two lemmas:

Lemma (3.2)

OID_1 *proves* $I_{\prec\alpha[n]} \subseteq I_{\prec\alpha}$.

Lemma (3.3)

Let $\eta(x, P)$ *be a positive arithmetical formula. Then* OID_1 *proves* $\eta(x, \lambda y.\,\exists\alpha\,(y \in I_\alpha)) \rightarrow \exists\alpha\,\eta(x, I_{\prec\alpha})$*.*

Functional interpretation of OID_1

Avigad and Towsner interprets OID_1 using a fragment $\mathsf{Q}_0\mathsf{T}_\Omega$ of the *classical* theory QT_Ω, obtained by restricting quantifiers to type ω only, and restricting atomic formulas to equations at type ω (thus omitting the predicate $\alpha = e$). The axioms of $\mathsf{Q}_0\mathsf{T}_\Omega$ are

(1) all equational machinery of T_Ω,

(2) the schema of induction on ω,

(3) the following rule schema of transfinite induction:

$$\frac{\theta(e) \qquad \alpha \neq e \wedge \forall n\, \theta(\alpha[n]) \to \theta(\alpha)}{\theta(t)}$$

Note that $\alpha \neq e$ can be understood as the formula $f(\alpha) = 1$ where f is defined by

$$f(e) = 0, \qquad f(\sup h) = 1.$$

The interpretation, continued

From a fixed instance of ID_1, we define a theory $\mathrm{Q_0T_\Omega} + (I)$ obtained from $\mathrm{Q_0T_\Omega}$ by adding a binary predicate $x \in I_\alpha$ and the following axioms:

(4) $\forall x\, x \notin I_e$,

(5) $\forall \alpha\, (\alpha \neq e \to \forall x\, (x \in I_\alpha \leftrightarrow \psi(x, I_{\prec\alpha})))$,

(6) $s \in I_\alpha \leftrightarrow t \in I_\beta$ whenever T_Ω proves $\alpha = \beta$ and $s = t$.

So $\mathrm{Q_0T_\Omega} + (I)$ combines a classical treatment of quantification over ω, with a constructive treatment of Ω.

To each formula φ in the language of OID_1 we associate a formula $\varphi^S \equiv \forall a\, \exists b\, \varphi_S(a, b)$, where the matrix $\varphi_S(a, b)$ is in the language of $Q_0\mathrm{T}_\Omega + (I)$.

$$(t \in I_\alpha)^S \equiv t \in I_\alpha$$
$$(s = t)^S \equiv s = t$$

The interpretation, continued

Suppose φ^S is $\forall a\,\exists b\,\varphi_S(a,b)$ and ψ^S is $\forall c\,\exists d\,\psi_S(c,d)$. Then we set:

$$\begin{aligned}
(\varphi \vee \psi)^S &:= \forall a, c\,\exists b, d\,(\varphi_S(a,b) \vee \psi_S(c,d)) \\
(\forall x\,\varphi)^S &:= \forall a\,\exists b\,(\forall x\,\varphi_S(a,b)) \\
(\forall \alpha\,\varphi)^S &:= \forall \alpha, a\,\exists b\,\varphi_S(a,b) \\
(\neg\varphi)^S &:= \forall B\,\exists a\,\big(\exists i\,\neg\varphi_S(a[i], B(a[i]))\big)
\end{aligned}$$

This interpretation has a monotonicity property:

if T_Ω proves $b \sqsubset b'$, then $\mathsf{Q}_0\mathsf{T}_\Omega + (I)$ proves $\varphi_S(a,b) \to \varphi_S(a,b')$

Theorem (4.2)

Suppose OID_1 *proves* φ, *and* φ^S *is the formula* $\forall a\,\exists b\,\varphi_S(a,b)$. *Then there are terms* b *of* T_Ω *involving at most the variables* a *and the free variables of* φ *of type* Ω *such that* $\mathsf{Q}_0\mathsf{T}_\Omega + (I)$ *proves* $\varphi_S(a,b)$.

Interpreting $\mathsf{Q_0T_\Omega} + (I)$ in $\mathsf{QT}^{\mathsf{i}}_\Omega$

Sieg provided a direct interpretation of ID_1 in $\mathsf{ID}_2^{\mathsf{i,sp}}$ by embedding an infinitary classical proof system. Using this method, Avigad and Towsner defines an infinitary proof system in $\mathsf{QT}^{\mathsf{i}}_\Omega$ in a way that preserves Π^0_2-sentences for proofs in $\mathsf{Q_0T_\Omega} + (I)$.

This works essentially because well-founded proof trees can be modelled by elements of Ω.

Theorem (5.7)

Every Π^0_2-theorem of $\mathsf{Q_0T_\Omega} + (I)$ is provable in $\mathsf{QT}^{\mathsf{i}}_\Omega$.

Application

As we mentioned, an application of functional interpretation is *proof mining* or *unwinding* – finding constructive content in classical proofs.

Avigad and Towsner applied the ideas behind their logical work to extract ordinal bounds in the Furstenberg-Zimmer structure theorem for measure preserving systems (this theorem in turns gives a perspicous proof of Szemerédi's theorem in combinatorics).

Furstenberg-Zimmer structure theorem

Theorem (Furstenberg-Zimmer structure theorem)

Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system. Then there exists an ordinal α and a factor $Y_\beta = (Y_\beta, \mathcal{Y}_\beta, \nu_\beta, S_\beta)$ for every $\beta \leq \alpha$ with the following properties:

1. *$Y_\emptyset$ is a point.*
2. *For every successor ordinal $\beta + 1 \leq \alpha$, $Y_{\beta+1}$ is a compact extension of Y_β.*
3. *For every limit ordinal $\beta \leq \alpha$, Y_β is the inverse limit of the Y_γ for the $\gamma < \beta$, in the sense that $L^2(Y_\beta)$ is the closure of $\bigcup_{\gamma<\beta} L^2(Y_\gamma)$.*
4. *X is a weakly mixing extension of Y_α.*

Using the logical apparatus

In fact, the application to Szemerédi's theorem doesn't require weak mixing, but only an approximation. Avigad and Towsner proves that the required property obtains fairly low in the Furstenberg-Zimmer tower, before $\omega^{\omega^{\omega}}$. But, as they also say:

> *It is interesting to note, however, that the logical considerations drop out of the final results. The metamathematical results provide a deeper understanding of the role that strong nonconstructive principles play in ordinary mathematical reasoning, and provide a guide to interpreting particular mathematical proofs in more explicit terms. But if one is only interested in the latter, at the end of the day, one is left with a purely mathematical proof.*

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