Proof theory of homotopy type theory: what we know so far

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Outline

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What rests on what?

- Hilbert's Program relativized: proof-theoretic reductions.
- Foundational reduction:
  - infinitary to finitary
  - impredicative to predicative
  - non-constructive to constructive

(Feferman, 1993)

Predicativity, historical background

- Poincaré and Russell formulated the vicious circle principle as a way to avoid the paradoxes.
- Zermelo (1908): *and up to now it has not occurred to anyone to regard this as something illogical* (referring to the least upper bound principle).
- Weyl (1918) in *das Kontinuum*: we only really need the least upper bound principle for *sequences*, not arbitrary subsets of $\mathbb{R}$.

The principle which enables us to avoid illegitimate totalities may be stated as follows: "Whatever involves all of a collection must not be one of the collection"; [...] *We shall call this the "vicious-circle principle," because it enables us to avoid the vicious circles involved in the assumption of illegitimate totalities. (Whitehead and Russell, 1910)*

Predicativity given the natural numbers

- Take the natural numbers for granted.
- Sets are constructed through predicative definition.
- How far can you go in this way? Ramified analysis (RA_α) where you can go to level α, if in a previously secured level you can prove α is well-founded.
- Feferman-Schütte analysis of predicativity: Γ_0.
- Generalized predicativity? (see below!)

The proof-theoretic ordinal of a theory T can be defined as:

$$|T| = \sup\{\mathrm{otyp}(\prec) \mid \prec \text{ is primitive recursive and } T \vdash \mathrm{TI}(\prec, X)\}$$

where $\mathrm{otyp}(\prec)$ is the order-type of the $\prec$ and $\mathrm{TI}(\prec, X)$ means that X (a free parameter) satisfies transfinite induction along $\prec$; this is the constructive way to say that $\prec$ is well-ordered.

As defined, this is a rather blunt invariant, but most calculations we know of in fact give much more precise information, e.g., about provably total recursive functions.

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A brief history

1971 First version of Martin-Löf type theory, later proved to be inconsistent by Girard.

1973 Martin-Löf introduces a predicative version of his type theory.

1977 Aczel proves $|\mathrm{ML}_1| = \varphi(\varepsilon_0, 0)$.

1979 Extensional Martin-Löf type theory (basis of NuPrl).

1982 Feferman and Jervell prove *Hancock's conjecture*: $|\mathrm{ML}_{<\omega}| = \Gamma_0$ (indep. by Aczel and Beeson).

1984 Intensional Martin-Löf type theory (basis of HoTT).

1992 Palmgren: interpreting iterated ID-systems into type theories.

1993 Setzer: $|\mathrm{ML}_1\mathrm{W}| = \psi_\Omega(\Omega_{I+\omega})$

1994 Griffor, Rathjen: $|\mathrm{ML}_1\mathrm{V}| = |\mathrm{ID}_1| = \psi_\Omega(\varepsilon_{\Omega+1})$, $\mathrm{ML}_1\mathrm{W}$ is a bit stronger than KPi.

1994 Griffor, Rathjen, Palmgren: MLQ slightly weaker than KPM.

1996 Setzer: $|\mathrm{MLM}| = \psi_\Omega(\Omega_{M+\omega})$.

1997 Rathjen: $|\mathrm{MLU}| = \Gamma_0$, $|\mathrm{MLS}| = \varphi(1, \Gamma_0, 0)$.

Kripke-Platek set theory

Very useful tool in proof theory, KP, with axioms:

(Ext') $(\forall u)(\forall v)[(\forall x \in u)(z \in v) \wedge (\forall x \in v)(x \in u) \to u = v]$.

(Pair), (Union), (Found) as usual.

(Δ_0-Sep) Bounded separation, as we saw yesterday:

$$(\forall \vec{v})(\forall u)(\exists z)\big[(\forall x \in z)(x \in u \wedge F(x, \vec{v})) \\ \wedge (\forall x \in u)(F(x, \vec{v}) \to x \in z)\big],$$

where $F(x, \vec{v})$ is a *bounded* (Δ_0) formula.

(Δ_0-Coll) For any bounded formula $F(x, y, \vec{v})$:

$$(\forall \vec{v})(\forall u)\big[(\forall x \in u)(\exists y)F(x, y, \vec{v}) \\ \to (\exists z)(\forall x \in u)(\exists y \in z)F(x, y, \vec{v})\big],$$

- A transitive set $\mathbb{A}$ is called *admissible* if $\mathbb{A} \models \mathrm{KP}$.
- An ordinal is called admissible if L_α is admissible, where

$$L_0 := \emptyset$$
$$L_{\alpha+1} := \mathrm{Def}(L_\alpha)$$
$$L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$$

  is Gödel's constructible hierarchy.
- An ordinal is called recursively inaccessible if it is admissible and a limit of admissibles.

Ordinal	ID-system	Set theory	Type theory
ω^ω	PRA		CPRC
ε_0	PA	$\mathrm{KP}\omega^0$	ML
Γ_0	$\widehat{\mathrm{ID}}_{<\omega}$	KPl^0	$\mathrm{ML}_{<\omega}$
$\varphi(1,\Gamma_0,0)$	$\widehat{\mathrm{ID}}_{<\Gamma_0}$		MLS
$\psi_\Omega(\varepsilon_{\Omega+1})$	ID_1	$\mathrm{KP}\omega$	$\mathrm{ML}_1\mathrm{V}$
$\psi_\Omega(\varepsilon_{\Omega_\omega+1})$	ID_ω	KPl	
$\psi_\Omega(\varepsilon_{I+1})$		KPi	$\mathrm{ML}_{1\mathrm{W}}$
$\psi_\Omega(\Omega_L)$		KPh	$\mathrm{ML}_{<\omega}\mathrm{W}$

Here, I is the first recursively inaccessible ordinal, while L is the limit of the first ω recursively inaccessible ordinals.

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For this talk, I want to define the HoTT variant of any Martin-Löf type theory as simply obtained by adding the Univalence Axiom (for all universes Type),

$$\Pi(A, B : \mathrm{Type}).\ \mathrm{isequiv}\big(\mathrm{idtoequiv} : (A =_{\mathrm{Type}} B) \to (A \simeq B)\big)$$

and a simple *higher inductive type* (HIT), the (homotopy) pushout, see below.

Also known as HITs. A simple example is the circle $\mathbb{S}^1$ as an ∞-groupoid freely generated by $\mathrm{base} : \mathbb{S}^1$ and $\mathrm{loop} : \mathrm{base} =_{\mathbb{S}^1} \mathrm{base}$.
For homotopy theory, one only uses certain finitary HITs, which all seem to be reducible to homotopy pushouts: if $f : C \to A$ and $g : C \to B$, then this is the type D generated by:

$$\begin{aligned} \mathrm{inl} &: A \to D \\ \mathrm{inr} &: B \to D \\ \mathrm{glue} &: (c : \mathrm{C}) \to \mathrm{inl}(f\,c) =_D \mathrm{inr}(g\,c) \end{aligned}$$

Reduction of other HITs to pushouts

See the Lean HoTT library for many instances, including:

- homotopy coequalizers, suspension, join, sequential homotopy colimits,
- classifying types of discrete groups, Rezk completion,
- propositional truncation (van Doorn),
- n-truncation (Rijke).

- The Cauchy reals in HoTT:

$$\begin{aligned} \mathrm{rat} &: \mathbb{Q} \to \mathbb{R} \\ \lim &: (x : \mathbb{Q}_+ \to \mathbb{R}) \to (\forall \delta, \varepsilon : \mathbb{Q}_+, x_\delta \sim_{\delta+\varepsilon} x_\varepsilon) \to \mathbb{R} \\ \mathrm{eq} &: (u, v : \mathbb{R}) \to (\forall \varepsilon : \mathbb{Q}_+, u \sim_\varepsilon v) \to u =_{\mathbb{R}} v \end{aligned}$$

  eliding clauses for $\sim_\varepsilon$, and set truncation.

- The cumulative hierarchy V:

$$\begin{aligned} \mathrm{set} &: (A : \mathrm{Type}) \to (f : A \to V) \to V \\ \mathrm{eq} &: (A, B : \mathrm{Type}) \to (f : A \to V) \to (g : B \to V) \\ &\to (\forall a : A, \exists b : B, f\,a =_V g\,b) \to (\forall b : B, \exists a : A, f\,a =_V g\,b) \\ &\to \mathrm{set}(A, f) =_V \mathrm{set}(B, g) \end{aligned}$$

Open problems

- (Voevodsky) Homotopy canonicity: if $\vdash t : \mathbb{N}$ is a closed term, then we can find a numeral n and a closed proof $\vdash p : t =_{\mathbb{N}} \underline{n}$.
- Prove that HoTT can be interpreted in any ∞-topos.
- Determine whether ∞-category theory can be adequately formalized in HoTT (if not, we need an extension).
- Facilitate reasoning with strict sets (such as $\mathbb{N}$).
- Formalize more abstract homotopy theory (e.g., recently Brunerie proved $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2\mathbb{Z}$; with Rijke and I did the H-space structure on $\mathbb{S}^3$ and real and complex projective spaces; what about H-space structure on $\mathbb{S}^7$ and quaternionic projective spaces?).
- General description of higher inductive types.
- Semantics of strict resizing rules.
- Computational interpretation of univalence and HITs
- *Today:* Proof-theoretic strength of univalence and HITs?

Conjecture

Conjecture: Univalence and HITs do not increase proof-theoretic strength.

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Due mainly to Voevodsky, written up by Kapulkin-LeFanu Lumsdaine (arXiv:1211.2851).

In the paper they work in ZFC plus two inaccessible cardinals. They solve the coherence issues by modeling a universe using *well-ordered* morphisms of simplicial sets. That's not necessary, can use the lifting universes approach of Hofmann-Streicher instead.

The model construction can then be formalized in KPl^0 or KPh (with W-types) and the model supports homotopy pushouts following unpublished work by Lumsdaine-Shulman.

There is at least some instances in which univalence and HITs do not increase proof-theoretic strength, namely for $\mathrm{ML}_{<\omega}$ and $\mathrm{ML}_{<\omega}\mathrm{W}$.

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	01	$01'$	$01\wedge$	$01\vee$	$01\wedge\vee$	$01\wedge\vee'$
w	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
we	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
wec	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$

The lower-right corner contains de Morgan, Kleene and Boolean algebras.

Add to that the Orton-Pitts interval theory, etc.

	01	$01'$	$01\wedge$	$01\vee$	$01\wedge\vee$	$01\wedge\vee'$
w	·	·	·	·	·	·
we	·	·	·	·	·	·
wec	·	·	·	·	·	…

The lower-right corner contains de Morgan, Kleene and Boolean algebras.

Add to that the Orton-Pitts interval theory, etc.

Theorem ((Grothendieck), B-Morehouse, Spitters)

Any of the notions of cubical sets in the table give rise to a test category. All except for the four in the top-left corner give strict test categories.

Th. Coquand, Bezem, Huber: *A model of type theory in cubical sets*, 2014.

Based on *symmetric* cubical sets with *uniform* Kan filling operations. An n-cube in an identity type $a =_A b$ is an $n+1$-cube in A: this only satisfies the computation rule for J *up to identity*.

Second cubical model

Cohen, Th. Coquand, Huber, Mörtberg: *Cubical Type Theory: a constructive interpretation of the univalence axiom*, 2015.

Based on *cartesian* cubical sets with connections and reversals satisfying de Morgan laws.

Again the computation rule for J is only propositional, but Andrew Swan devised a variation of the identity which satisfies the computation rule definitionally.

Current work by Simon Huber on proving weak normalization.

The model is fully constructive and can readily be formalized in a suitable constructive set theory, for instance $\mathrm{CZF}^- + \mathrm{Inac}$ (which has strength Γ_0 by Crosilla-Rathjen) or $\mathrm{CZF} + \mathrm{Inac}$.

In fact, Mark Bickford has formalized the model, and the syntax in the model (including a cumulative hierarchy of universes), in Nuprl. Inspecting his proof, we see it fits in $\mathrm{ML}_{<\omega}$, resp. $\mathrm{ML}_{<\omega}\mathrm{W}$.

We still have finitary HITs such as pushouts.

Conclusion: Not only, does univalence and simple HITs don't raise proof-theoretic strength over $\mathrm{ML}_{<\omega}$(W), but we have an interpretation into the corresponding non-HoTT systems.

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The NuPrl model: Constable's group at Cornell, in development since early 1980s, based on Martin-Löf 1979 type theory.

More explicitly, one defines an untyped programming language and *defines* the type theoretic judgments: what it means to be a type and what it means to be an element of a type. Then one verifies the rules of 1979 ML type theory.

They added other type formers, e.g., intersection types, partial function types, squash types, quotients, subsets, etc.

- Angiuli, Harper, Wilson: Computational Higher Type Theory I: Abstract Cubical Realizability, 2016
- Angiuli, Harper: Computational Higher Type Theory II: Dependent Cubical Realizability, 2016

Same principle, but work with untyped programming language with *dimension names* (corresponding to cartesian cubical sets) and a variation of the uniform Kan operations.

Much more complicated definitions of type theory judgments. Immediately extract canonicity result. So far no universe.

Formalizing computational higher type theory?

The natural place to formalize Harper et al.s work would be in Feferman's theories of explicit mathematics: an abstract operational setting (PCA) plus "types" given by elementary comprehension, join, and universes.

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Conclusions

- As Bas mentioned, we know what happens if we add AC: correlates with ZFC plus strongly inaccessible cardinals.
- Known results imply that univalence+simple HITs do not raise proof-theoretic strength wrt $\mathrm{ML}_{<\omega}$(W), and Bickford's formalization gives interpretation.
- Can we give a cubical model/type theory in which J computes for the natural path type?
- What is the strength of type theory with Prop? With propositional resizing?
- We still need to analyze more complicated HITs such as the Cauchy reals.

Foundational matters

- Take seriously a pluralist stance: we want several foundational systems and we want to calibrate strength of subsystems of each.
- Perhaps a finitist system is the best foundation from an ecumenical formalization perspective: formalize results of the form $T \vdash \varphi$ for various T as well as reductions.
- Is it a problem (from a foundational perspective) for type theory that the its judgments are not primitive recursively checkable?
- Is there a good way to give primitive recursively checkable evidence for the judgments of type theory?