The structure of Parafermion VOAs associated to $osp(1|2m)$

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Recall

$\mathfrak{g}$ - f.d. simple Lie alg $/\mathbb{C}$

$\mathfrak{h}$ — Cartan subalg.

Δ ~ root system

$(\cdot|\cdot)$ normalized $(\alpha|\alpha)=2$
for $\alpha \in \Delta^L$.

For $k \in \mathbb{Z}_{\geq 1}$, let

$V^k(\mathfrak{g})$ — the universal
affine VOA

$L_k(\mathfrak{g})$ — the simple

$M_{\mathfrak{h}}(k,0)$ — the Heisenberg subVOA of both.

Then

$$N(\mathfrak{g},k) = C(M_{\mathfrak{h}}(k,0), V^k(\mathfrak{g}))$$

$$K(\mathfrak{g},k) = C(M_{\mathfrak{h}}(k,0), L_k(\mathfrak{g}))$$

the simple ones.

the universal parafermion VOAs

The structure of $\mathcal{N}(\mathfrak{sl}_2, k)$ and $\mathcal{K}(\mathfrak{sl}_2, k)$

[Dong - Lam - Yamada, 09]

[Dong - Lam - Wang - Yamada, 09]

$\mathcal{N}(\mathfrak{sl}_2, k)$ is generated by ω, W^3, and strongly generated by ω, W^3, W^4. W^5 with concrete expressions of them

Also [Linshaw, 18] from

the point view of

$W(c, \lambda)$

C_2-cofiniteness and repres

of $K(\mathfrak{sl}_2, k)$

[Dong-Lam-Yamada, of]

[Arakawa-Lam-Yamada,

12, 17], [Dong-Wang,

16]

For general $\mathfrak{g}$, and $\alpha \in \Delta$

Let $\mathfrak{g}_\alpha$ be such that

$$\cong \mathfrak{sl}_2, \quad k_\alpha = \frac{2}{(\alpha|\alpha)} k.$$

Then [Dong-Wang, 10]

$N(\mathfrak{g}, k)$ $(\underline{K(\mathfrak{g}}, k))$ is

generated by $N(\mathfrak{g}_\alpha, k_\alpha)$

$(K(\mathfrak{g}_\alpha, k_\alpha))$, $\alpha \in \Delta_+$

[Dong-Wang, 11], [Dong-Ren, 17]

[Lam, 14], [J-Lin, 15]

$K(\mathfrak{sl}_n, 2)$

[Ai-Dong-Jiao-Ren, 18]

[Creutzig - Kanada - Linshaw, 17].

$\mathfrak{osp}(1|2m)$?

(1) $\mathfrak{osp}(1|2m)$ shares the same Cartan subalg with even part.

(2) Claimed by [Gorelik - Kac, 07] If $\mathfrak{g}$ is not a Lie alg. Then

$L_k(\mathfrak{g})$ is C_2-cofinite

iff $k \in \mathbb{Z}_{\geq 0}$ and $\mathfrak{g} = \mathfrak{osp}(1|2m)$

proved by [Ai-Lin, 21]

and [Creutzig-Linshaw, 21]

rationality.

Representations of $L_k(\mathfrak{osp}(1|2n))$

[Ramallo-de Santos, 90]

[Adamovic, 18]

$K(\mathfrak{osp}(1|2n), k)$, for k-admissible

[Creutzig - Fröhlich - Kanade, 17]

[Creutzig - Kanade - Liu - Ridout, 18].

[Kac, ?]

$N(osp(1|2n), k)$ and

$K(osp(1|2n), k)$. $k \in \mathbb{Z}_{\geq 1}$

Let $\mathfrak{g} = osp(1|2n)$.

$\mathfrak{h}$ - Cartan subalg.

Δ_0 - the root system of even

Δ_1 — the root system of odd

$\left(\Delta_1 = \frac{1}{2} \Delta_0^L \right)$

Δ_0^L — long roots of even part.

Δ_0^S — short roots of odd.

$(\cdot|\cdot)$ — normalized

$$(\alpha|\alpha) = 2, \quad \alpha \in \Delta_0^L$$

For $\alpha \in \Delta_{0(+)}^S$, take

$$\mathfrak{g}_\alpha = \mathbb{C} e_\alpha \oplus \mathbb{C} e_{-\alpha} \oplus \mathbb{C} h_\alpha.$$

$$h_\alpha = 2t_\alpha, \quad t_\alpha(h) = \alpha(h)$$

$$\mathfrak{g}_\alpha \cong \mathfrak{sl}_2(\mathbb{C})$$

For $\alpha \in \Delta_0^\perp$, let

$$\mathfrak{g}_\alpha = \mathbb{C}e_\alpha \oplus \mathbb{C}e_{-\alpha} \oplus \mathbb{C}h_\alpha,$$

$$h_\alpha = t_\alpha.$$

s.t. $\mathfrak{g}_\alpha \cong \mathfrak{sl}_2(\mathbb{C})$

and take $x_{\pm\frac{1}{2}\alpha} \in \mathfrak{g}_{\pm\frac{1}{2}\alpha}$ s.t.

$$[h_\alpha \ \ x_{\pm\frac{1}{2}\alpha}] = \pm x_{\pm\frac{1}{2}\alpha}$$

$$[e_\alpha, x_{-\frac{1}{2}\alpha}] = -x_{\frac{1}{2}\alpha}$$

$$[e_{-\alpha}, x_{\frac{1}{2}\alpha}] = -x_{-\frac{1}{2}\alpha}.$$

and

$$\{x_{\frac{1}{2}\alpha}, x_{\frac{1}{2}\alpha}\} = 2e_\alpha,$$

$$\{x_{-\frac{1}{2}\alpha}, x_{-\frac{1}{2}\alpha}\} = -2e_{-\alpha}$$

$$\{x_{\frac{1}{2}\alpha}, x_{-\frac{1}{2}\alpha}\} = h_\alpha.$$

Then

$$\bar{\mathfrak{g}}^\alpha = \mathbb{C}e_\alpha \oplus \mathbb{C}e_{-\alpha} \oplus \mathbb{C}h_\alpha$$

$$\oplus\, \mathbb{C}x_{\frac{1}{2}\alpha} \oplus \mathbb{C}x_{-\frac{1}{2}\alpha}$$

$$\simeq \mathfrak{osp}(1|2).$$

$V^k(\mathfrak{g})$ – the universal VOSA $k \in \mathbb{Z}_{\geq 1}$.

$k \in \mathbb{C}$. $k \neq -(n+\frac{1}{2})$

$$\omega_{aff} = \frac{1}{2k+2n+1}\Big[\sum_{i=1}^{n} h_i(-1) h_i(-1)\mathbb{1} + \sum_{\alpha \in \Delta_0} \frac{(\alpha|\alpha)}{2} e_\alpha(-1) e_{-\alpha}(-1)\mathbb{1} - \frac{1}{2} \sum_{\alpha \in \Delta_1(+)} \big(x_\alpha(-1) x_{-\alpha}(-1)\mathbb{1} - x_{-\alpha}(-1) x_\alpha(-1)\mathbb{1} \big) \Big].$$

[Kac–Kwon–Wakimoto, 03]

View $V^k(\mathfrak{g})$ as an $\mathfrak{h}$-module:

we have

$$V^k(\mathfrak{g}) = \bigoplus_{\lambda} V^k(\mathfrak{g})(\lambda),$$

where

$$V^k(\mathfrak{g})(\lambda) = \{ v \in V^k(\mathfrak{g}) \mid h \cdot v = \lambda(h) v, \ h \in \mathfrak{h} \}$$

$$\Rightarrow \quad V^k(\mathfrak{g})(\lambda) = M_{\hat{\mathfrak{h}}}(k, 0) \otimes N_\lambda,$$

$$N_\lambda = \{ v \in V^k(\mathfrak{g}) \mid h(m) v = \delta_{m,0} \lambda(h) v, \ m \geq 0 \}$$

$N_0 = N(\mathfrak{osp}(1|2n), k)$

Proposition For $k \in \mathbb{Z}_{\geq 1}$, $V^k(\mathfrak{g})(0)$ is generated by $\alpha_i(-1)\mathbb{1}$,

$e_\alpha(-2)e_\alpha(-1)\mathbb{1}$, $x_{-\frac{1}{2}\alpha}(-1)x_{-\frac{1}{2}\alpha}(-1)\mathbb{1}$,

for $\alpha \in \Delta^L_{0(+)}$, and

$e_{-\beta}(-2)e_\beta(-1)\mathbb{1}$, $\beta \in \Delta^S_{0(+)}$.

For $\alpha \in \Delta_{0(+)}$, Let w^α, W^α

be the generators of

$K(\mathfrak{g}_\alpha, k_\alpha)$, $\mathfrak{g}_\alpha \cong \mathfrak{sl}_2(\mathbb{C})$

and Let

$$\bar{\omega}_\alpha = -h_\alpha(-1)^2\mathbb{1} + 4k\, x_{\frac{1}{2}\alpha}(-1)\, x_{-\frac{1}{2}\alpha}(-1)\mathbb{1}$$
$$-2k\, h_\alpha(-2)\mathbb{1}$$

$$\bar{W}^3_\alpha = k^2 h_\alpha(-3)\mathbb{1} + 3k\, h_\alpha(-2)\, h_\alpha(-1)\mathbb{1}$$
$$+ h_\alpha(-1)^3\mathbb{1} - 6k\, h_\alpha(-1)\, x_{\frac{1}{2}\alpha}(-1)\, x_{-\frac{1}{2}\alpha}(-1)\mathbb{1}$$
$$+ 6k^2 x_{\frac{1}{2}\alpha}(-2)\, x_{-\frac{1}{2}\alpha}(-1)\mathbb{1} - 6k^2 x_{\frac{1}{2}\alpha}(-1)\, x_{-\frac{1}{2}\alpha}(-2)\mathbb{1},$$

for $\alpha \in \Delta^{L}_{0(+)}$.

Let J be the maximal ideal of $V^k(\mathfrak{g})$

Recall from [Gorelik–Serganova, (8)]

$$J = \langle e_\theta(-1)^{k+1}\mathbb{1} \rangle.$$

$$\mathfrak{g} = \mathfrak{osp}(1|2n).$$

$$J = \bigoplus_\lambda M_{\mathfrak{h}}(k,\lambda) \otimes (J \cap N_\lambda)$$

$\Rightarrow I = J \cap N(\mathfrak{g}, k)$.

We have

$$I = N(\mathfrak{g}, k)\, \underline{e_\theta(-1)^{k+1}}\, e_\theta(-1)^{k+1} \mathbb{1}$$

For $\alpha \in \Delta^L_{0(+)}$, let P_α be the subalg of $K(\mathfrak{g}, k)$ generated by ω_α, W_α^3, $\bar{\omega}_\alpha$ $\overline{W}_\alpha^3$. Then

Thm 1) For $\alpha \in \overset{L}{\Delta}_{0(+1)}$, P_α is simple and $\cong K(osp(1|2), k)$

2) $K(osp(1|2n), k)$ is generated by P_α, $\alpha \in \overset{L}{\Delta}_{0(+1)}$, and

$K(\mathfrak{g}_\alpha, k_\alpha)$, $\alpha \in \overset{S}{\Delta}_{0(+)}$. $\mathfrak{g}_\alpha \cong \underline{sl_2}$.

Remark 1) We do not give the strong generators.

2) $Aut(K(osp(1|2), k)) \cong \mathbb{Z}_2 = \langle \sigma \rangle$

σ comes from the even part of $osp(1|2)$

3) It seems that the structure of the parafermion VOAS are quite different.

Thanks.

Theorem (J – Wang): Let k be a positive integer, then the universal affine parafermion VOA $N(osp(1|2n), k)$ is generated by ω_α, $\bar{\omega}_\alpha$, W^3_α, $\bar{W}^3_\alpha$ for $\alpha \in \Delta^L_{0(+)}$ and ω_β, W^3_β for $\beta \in \Delta^S_{0(+)}$.

$K(sl_2, \mathbb{C})$

That is, $N(osp(1|2n), k)$ is generated by $N(osp(1|2), k)$ for $\alpha \in \Delta^L_{0(+)}$ and $N(sl_2, k_\alpha)$ for $\alpha \in \Delta^S_{0(+)}$, $k_\alpha = \frac{2}{(\alpha|\alpha)} k$.

$osp(1|2n)_0$ $\quad C$