Continuous Functions

A function f : ℝ → ℝ is continuous at x if for all sequences (x_n) with lim_{n→∞} x_n=x we have lim_{n→∞} f(x_n) = f(x).

> $limit\left(\sin(x), x = \frac{\pi}{2}\right)$

$$1 \quad (1.1)$$

> $\sin\left(\frac{\pi}{2}\right)$

$$1 \quad (1.2)$$

Attention: Checking just for one sequence may not be enough.

> $limit\left(\frac{\pi}{2} + \frac{1}{n}, n = \infty\right)$

$$\frac{1}{2}\,\mathrm{Pi} \quad (1.3)$$

> $limit\left(\sin\left(\frac{\pi}{2} + \frac{1}{n}\right), n = \infty\right)$

$$1 \quad (1.4)$$

Another example: The sign function.

> $\mathrm{signum}\left(-\frac{\pi}{2}\right), \mathrm{signum}(0), \mathrm{signum}(1000)$

$$-1, 0, 1 \quad (1.5)$$

The sign function is continuous at x=1.

> $limit(\mathrm{signum}(x), x = 1)$

$$1 \quad (1.6)$$

The sign function is **not** continuous at x=0 ...

> $limit(\mathrm{signum}(x), x = 0)$

$$undefined \quad (1.7)$$

> $limit\left(\mathrm{signum}\left(\frac{1}{n}\right), n = \infty\right)$

$$1 \quad (1.8)$$

> $limit\left(\mathrm{signum}\left(-\frac{1}{n}\right), n = \infty\right)$

$$-1 \quad (1.9)$$

Functions defined via "proc" are not eligible for limit computations.

> $f := \mathbf{proc}(x)\ \mathbf{if}\ (x \le 0)\ \mathbf{then}\ 0\ \mathbf{else}\ 1\ \mathbf{end\ if};\ \mathbf{end\ proc};$

$$f := \mathbf{proc}(x)\ \mathbf{if}\ x <= 0\ \mathbf{then}\ 0\ \mathbf{else}\ 1\ \mathbf{end\ if\ end\ proc} \quad (1.10)$$

> $f(0), f\left(evalf\left(\frac{\pi}{2}\right)\right), f(100)$

$$0, 1, 1 \quad (1.11)$$

> $limit(f(x), x = 0)$

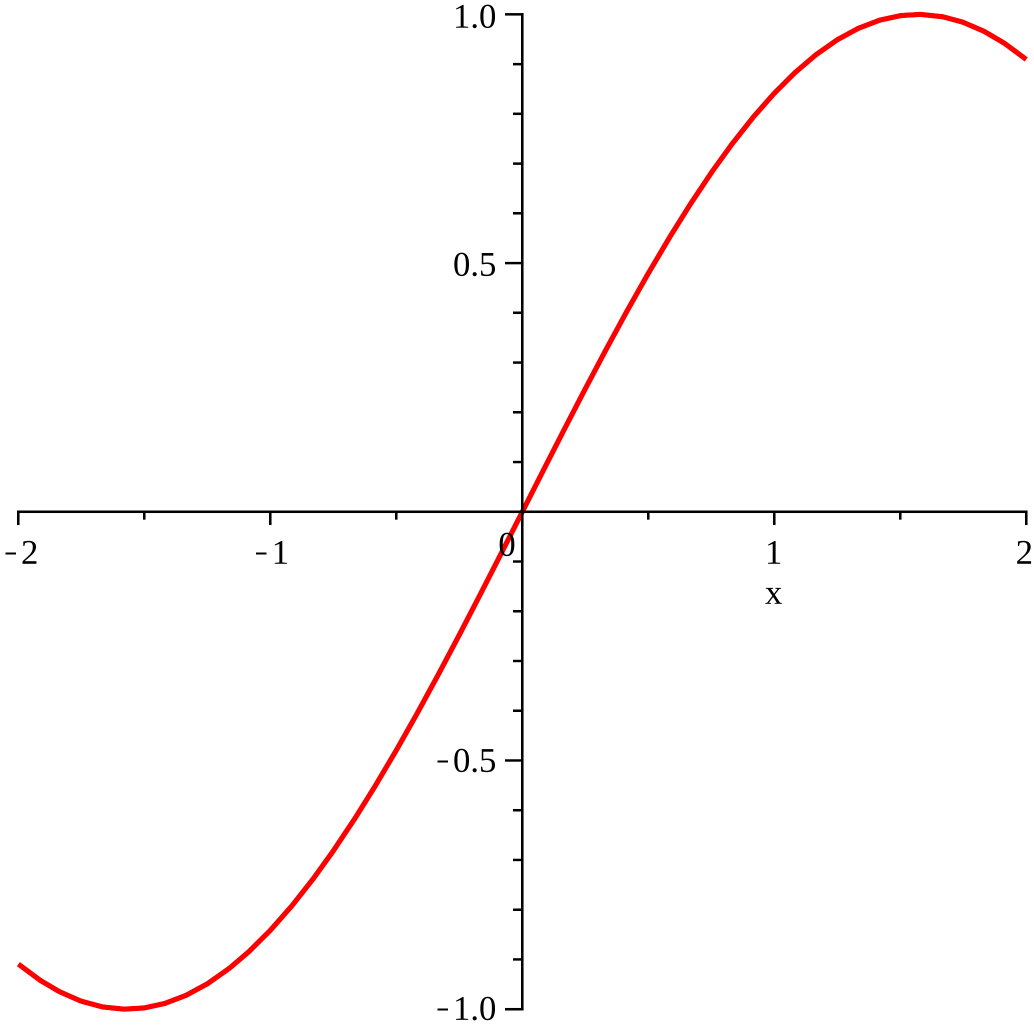
```
Error, (in f) cannot determine if this expression is true
or false: x <= 0
```

Intermediate Value Theorem

Let f : ℝ→ℝ be continuous. For all a,b,c in ℝ with a<b<c such that there are x,z in ℝ with f(x)=a

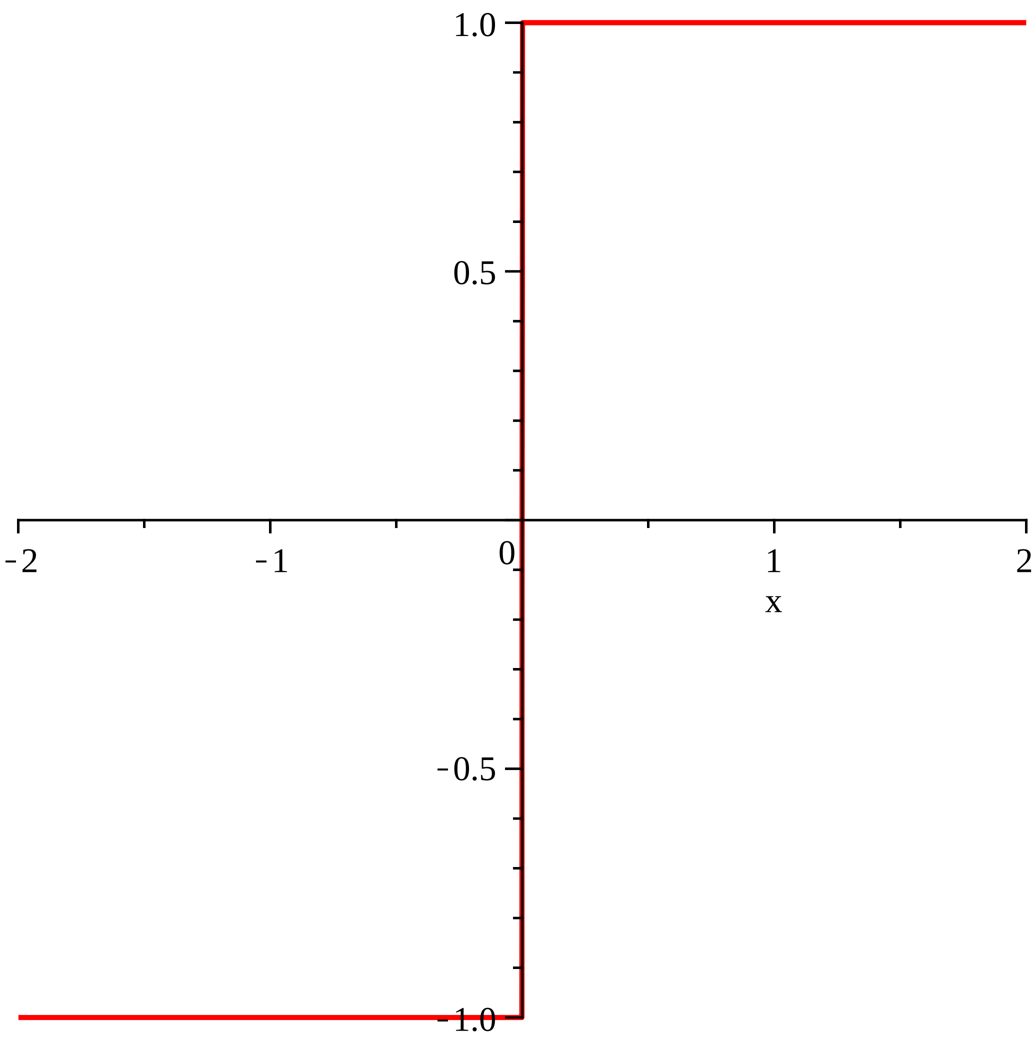
and f(z)=c there exists a y in ℝ with f(y)=b.

> $plot(\sin(x), x=-2..2, thickness=2)$



Plotting non-continuous functions is difficult. Notice the (erroneous) vertical line segment on the ordinate axis.

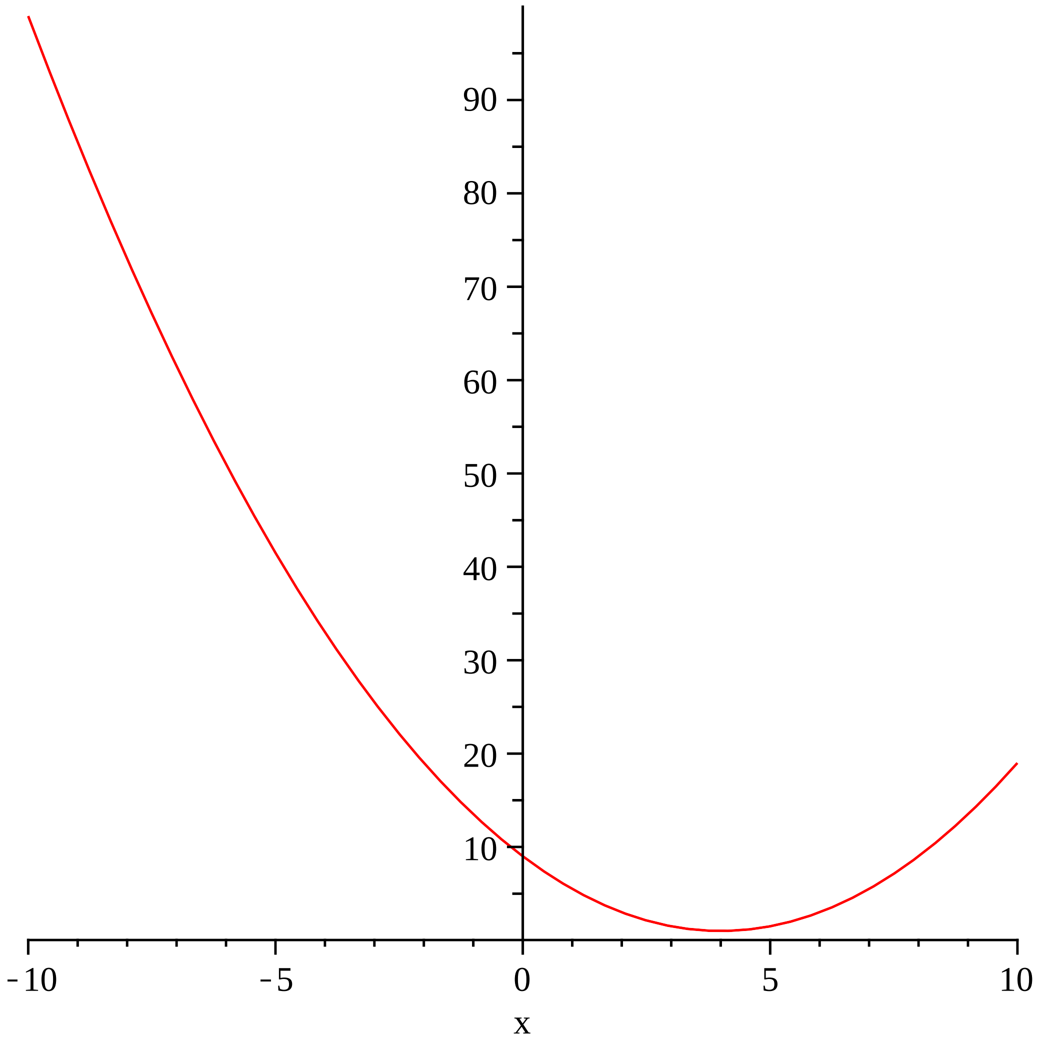
> $plot(\mathrm{signum}(x), x=-2..2, thickness=2)$



Plotting Polynomials (of High Degree)

Plotting a polynomial function (of low degree).

> $plot\left(\frac{1}{2}x^2 - 4 \cdot x + 9, x = -10..10\right)$



Let us look at a polynomial of degree 8.

> $r := 9\,x^4 - 8\,x^2 + 2 - 3\,x^6 + x^8$

$$r := 9\,x^4 - 8\,x^2 + 2 - 3\,x^6 + x^8 \quad \textbf{(2.1.1)}$$

It has four real zeroes.

> $fsolve(r = 0)$

$$-.8281056683,\ -.6655511224,\ 0.6655511224,\ 0.8281056683 \quad \textbf{(2.1.2)}$$

> $plot(r, y = -0.5\,..0.5, view = [-1.2\,..1.2, -0.5\,..2.3])$

```
Warning, unable to evaluate the function to numeric
values in the region; see the plotting command's help
page to ensure the calling sequence is correct
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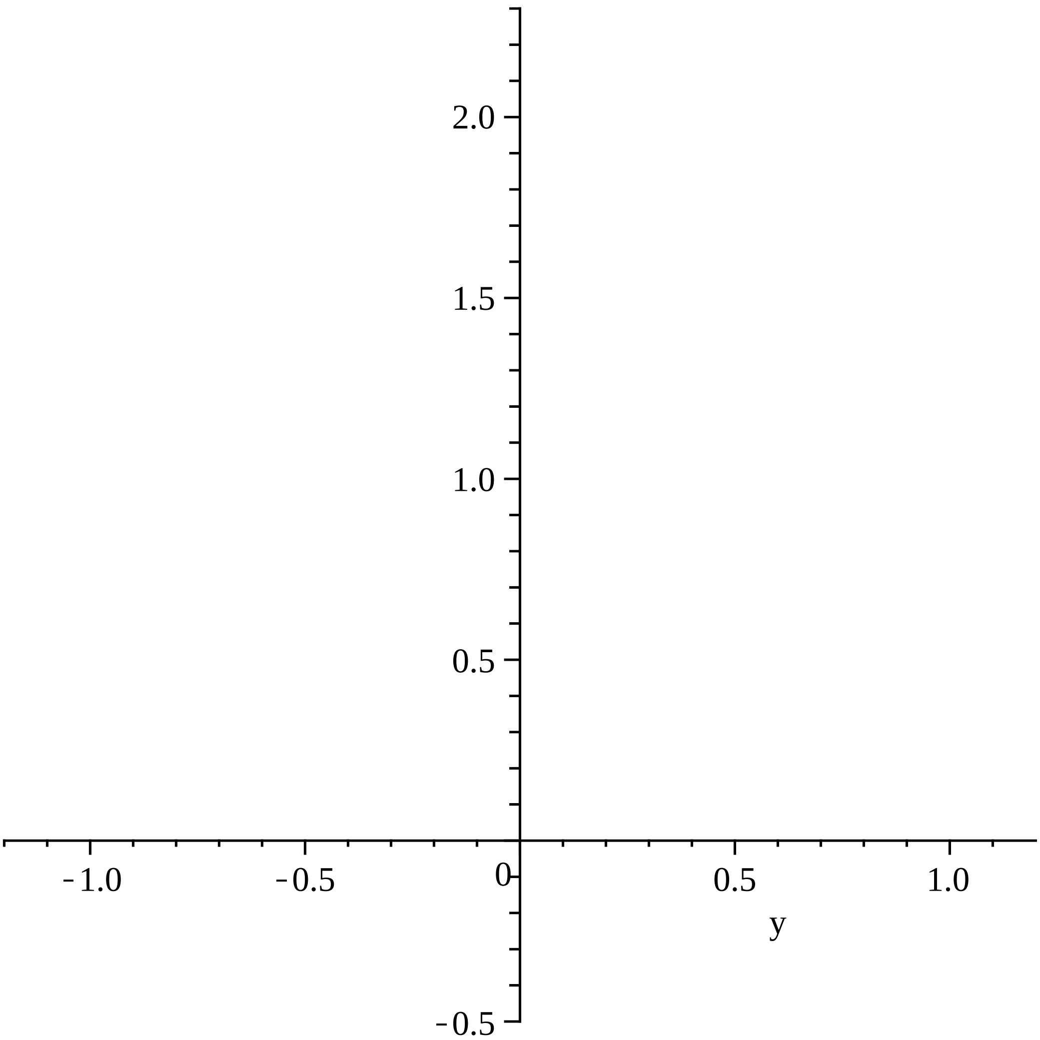
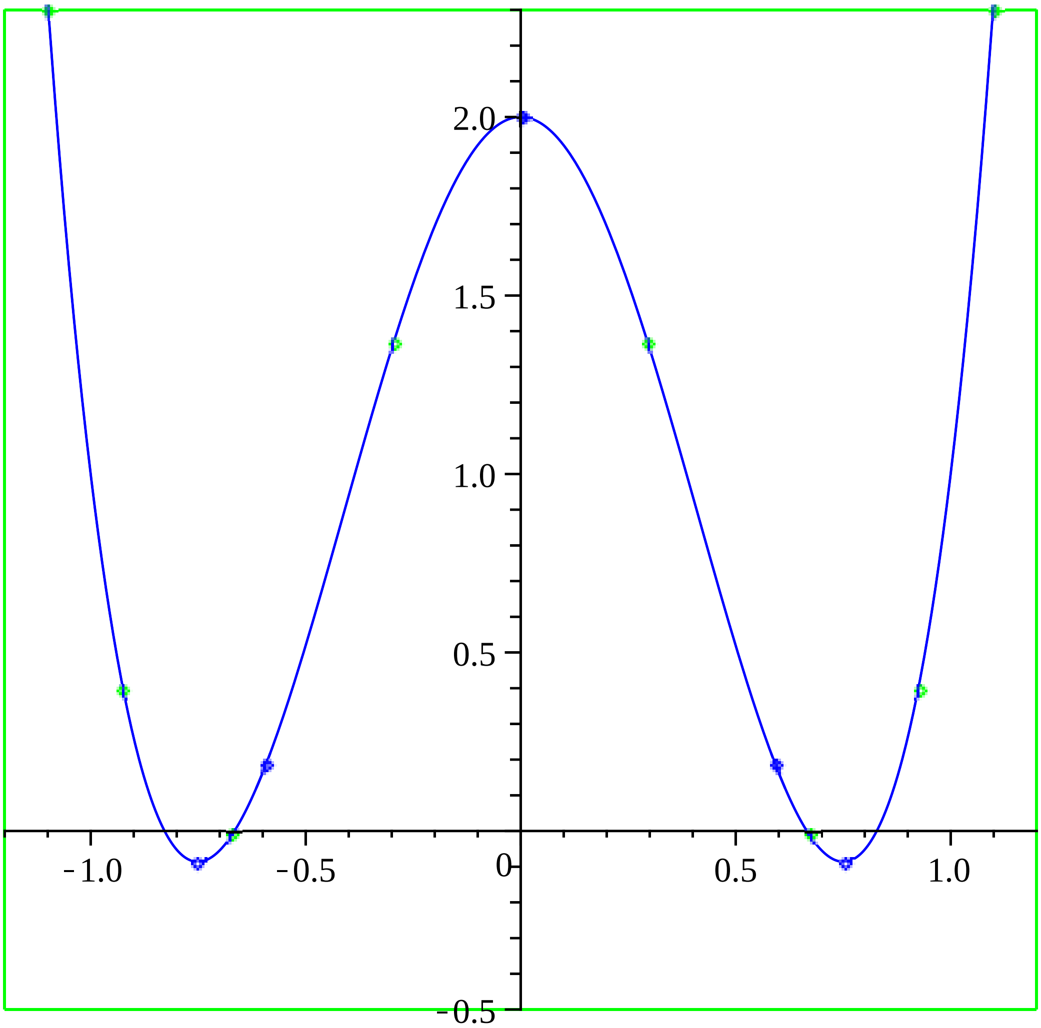


There are specific Maple functions for plotting polynomials and, more generally, (real) algebraic curves.

> $with(algcurves) : with(plots) :$

The curve is defined as the { (x,y) : y-r=0 }. Note that r depends on x.

> $plot\_real\_curve(y-r, x, y, view=[-1.2..1.2, -0.5..2.3])$



Finding Zeroes

A formula known from high school.

> $solve(x^2 + a \cdot x + b = 0, x)$

$$-\frac{1}{2}a + \frac{1}{2}\sqrt{a^2 - 4b}, -\frac{1}{2}a - \frac{1}{2}\sqrt{a^2 - 4b} \quad \textbf{(2.2.1)}$$

A similar formula for polynomials of degree 3.

> $solve(x^3 + a \cdot x^2 + b \cdot x + c = 0, x)$

$$\frac{1}{6}\left(36\,b\,a - 108\,c - 8\,a^3 + 12\sqrt{12\,b^3 - 3\,b^2\,a^2 - 54\,b\,a\,c + 81\,c^2 + 12\,c\,a^3}\right)^{1/3}$$

$$-\left(6\left(\frac{1}{3}b - \frac{1}{9}a^2\right)\right) \Big/ \left(36\,b\,a - 108\,c - 8\,a^3 + 12\sqrt{12\,b^3 - 3\,b^2\,a^2 - 54\,b\,a\,c + 81\,c^2 + 12\right.$$

$$^{/3} - \frac{1}{3}a, -\frac{1}{12}\left(36\,b\,a - 108\,c - 8\,a^3\right.$$

$$\left. + 12\sqrt{12\,b^3 - 3\,b^2\,a^2 - 54\,b\,a\,c + 81\,c^2 + 12\,c\,a^3}\right)^{1/3}$$

$$+\left(3\left(\frac{1}{3}b - \frac{1}{9}a^2\right)\right) \Big/ \left(36\,b\,a - 108\,c - 8\,a^3 + 12\sqrt{12\,b^3 - 3\,b^2\,a^2 - 54\,b\,a\,c + 81\,c^2 + 12\right.$$

$$^{/3} - \frac{1}{3} a + \frac{1}{2} I \sqrt{3} \left(\frac{1}{6} \left(36 b a - 108 c - 8 a^3 \right.\right.$$

$$\left. + 12 \sqrt{12 b^3 - 3 b^2 a^2 - 54 b a c + 81 c^2 + 12 c a^3} \right)^{1/3}$$

$$+ \left(6 \left(\frac{1}{3} b - \frac{1}{9} a^2\right)\right) \Big/ \left(36 b a - 108 c - 8 a^3 + 12 \sqrt{12 b^3 - 3 b^2 a^2 - 54 b a c + 81 c^2 + 12\right.$$

$$\left.^{/3}\right), -\frac{1}{12} \left(36 b a - 108 c - 8 a^3\right.$$

$$\left. + 12 \sqrt{12 b^3 - 3 b^2 a^2 - 54 b a c + 81 c^2 + 12 c a^3} \right)^{1/3}$$

$$+ \left(3 \left(\frac{1}{3} b - \frac{1}{9} a^2\right)\right) \Big/ \left(36 b a - 108 c - 8 a^3 + 12 \sqrt{12 b^3 - 3 b^2 a^2 - 54 b a c + 81 c^2 + 12\right.$$

$$^{/3} - \frac{1}{3} a - \frac{1}{2} I \sqrt{3} \left(\frac{1}{6} \left(36 b a - 108 c - 8 a^3 \right.\right.$$

$$\left. + 12 \sqrt{12 b^3 - 3 b^2 a^2 - 54 b a c + 81 c^2 + 12 c a^3} \right)^{1/3}$$

$$+ \left(6 \left(\frac{1}{3} b - \frac{1}{9} a^2\right)\right) \Big/ \left(36 b a - 108 c - 8 a^3 + 12 \sqrt{12 b^3 - 3 b^2 a^2 - 54 b a c + 81 c^2 + 12\right.$$

$$\left.^{/3}\right)$$

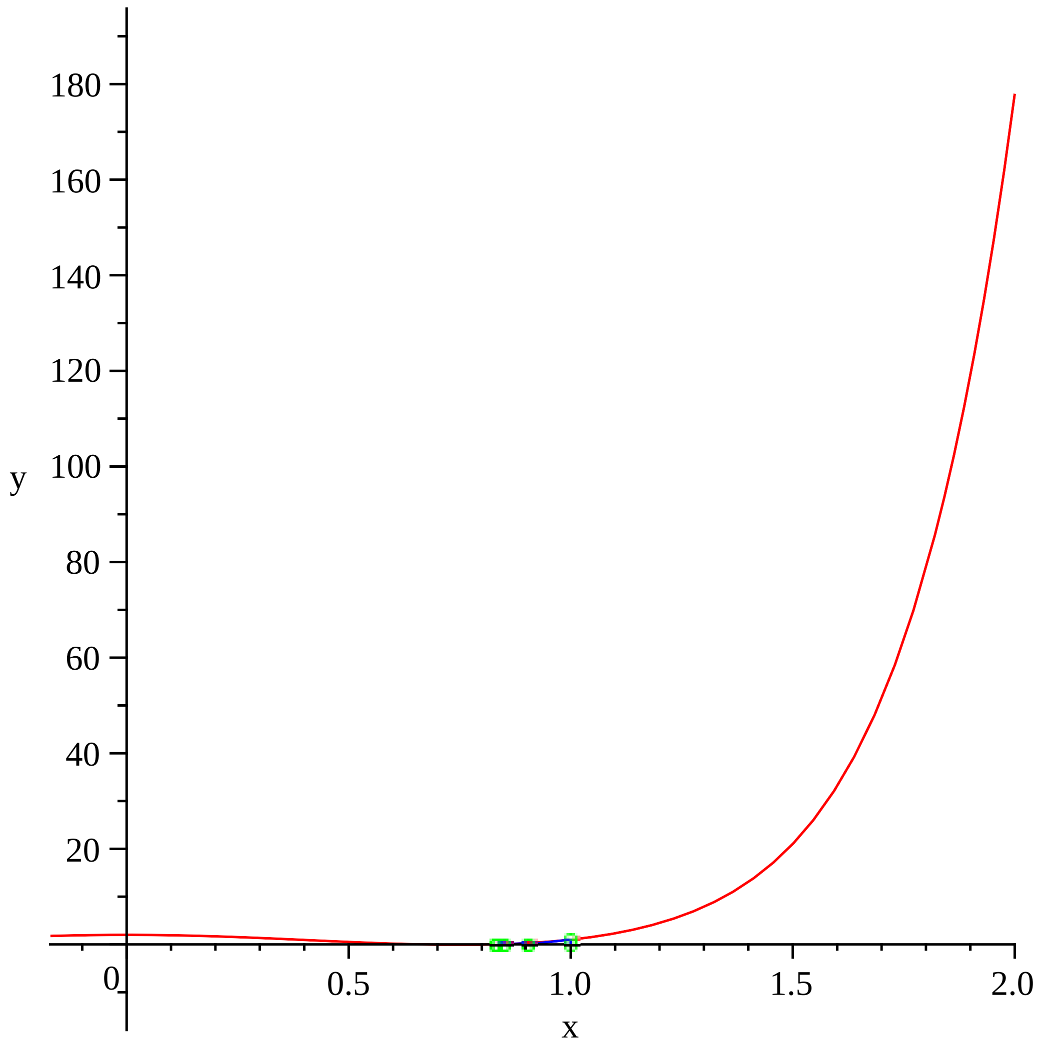
There is also one for degree 4, but Maple does not know about it.

> $solve(x^4 + a \cdot x^3 + b \cdot x^2 + c \cdot x + d = 0, x)$

$$\mathrm{RootOf}\left(\_Z^4 + a\,\_Z^3 + b\,\_Z^2 + c\,\_Z + d\right) \qquad \textbf{(2.2.3)}$$

Starting with degree 5 things get considerably more difficult. But still numerical solutions can be found.

> $with(Student[Calculus1]):$
> $NewtonsMethodTutor(r);$



Random Sampling

> $MySetOracle := \textbf{proc}(p)$
> $\textbf{local}\ x, y;\ x := p[1];\ y := p[2];$
> $\textbf{if}\ (x < 0)\ \textbf{or}\ (x > 1)\ \textbf{or}\ (y < 0)\ \textbf{or}\ (y > 1)\ \textbf{then return}\ \mathit{false};\ \textbf{end if};$
> $\textbf{if}\ \left(y > \frac{1}{2} \cdot \sin(\mathit{evalf}(x))\right)\ \textbf{then return}\ \mathit{false};\ \textbf{end if};$
> $\textbf{if}\ \left((x-0.5)^2 + (y-0.5)^2 < \frac{1}{10}\right)\ \textbf{then return}\ \mathit{false};\ \textbf{end if};$
> $\textbf{return}\ \mathit{true};$
> $\textbf{end proc}:$

> $MySetOracle([1, 1])$

$$\mathit{false} \quad (3.1)$$

> $MySetOracle([0.2, 0])$

$$\mathit{true} \quad (3.2)$$

> $MyRandomSource := \frac{rand(0..1000)}{1000}:$

> $MyRandomSource(\ ), MyRandomSource(\ )$

$$\frac{533}{1000}, \frac{803}{1000} \quad (3.3)$$

> $MyRandomPoint := (\ ) \rightarrow [MyRandomSource(\ ), MyRandomSource(\ )];$

$$MyRandomPoint := (\ ) \rightarrow [MyRandomSource(\ ), MyRandomSource(\ )] \quad (3.4)$$

> *MyRandomPoint*( )

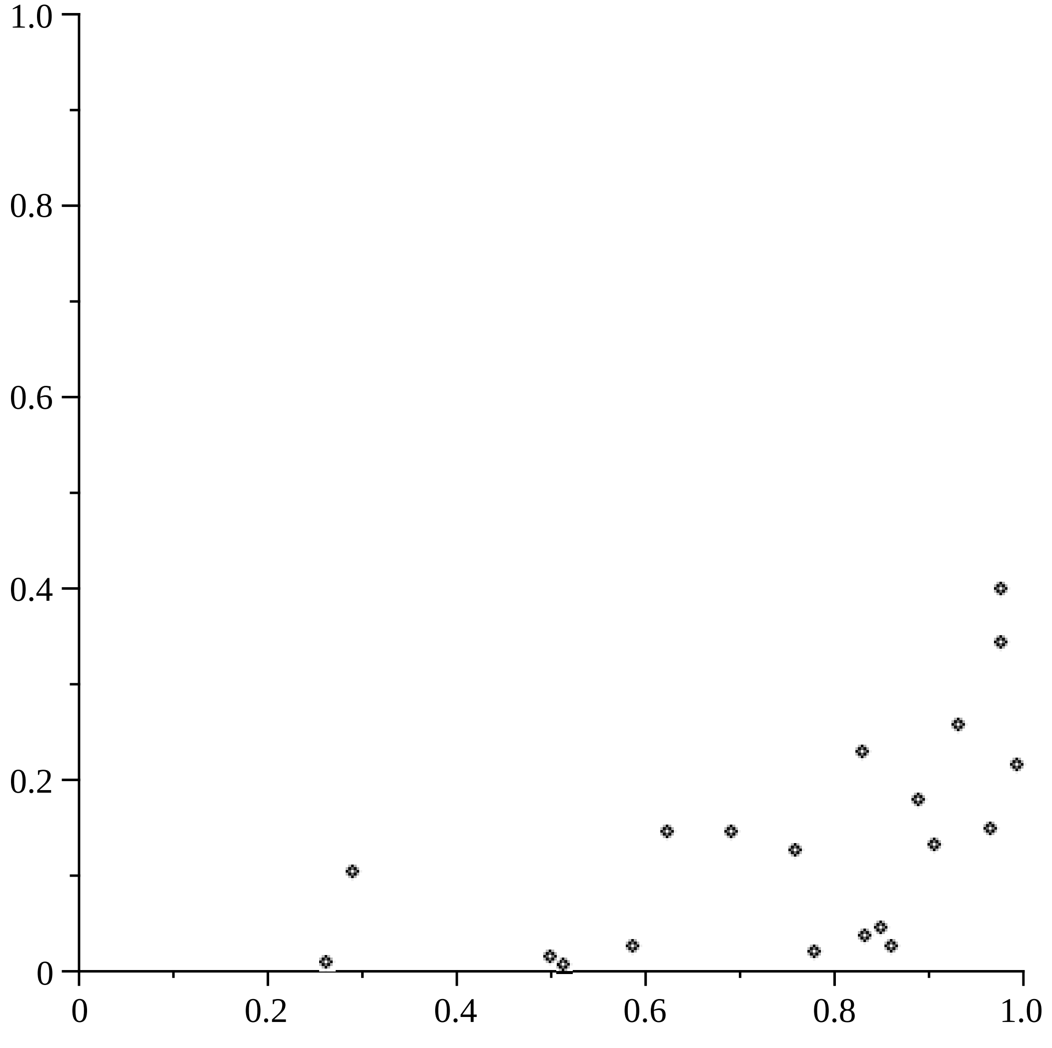
$$\left[\frac{921}{1000}, \frac{52}{125}\right] \quad (3.5)$$

```
> RandomPointsInSet := proc(SetOracle, n)
   local k, Points, p; Points := [ ];
   for k from 1 to n do
     p := MyRandomPoint( );
     if MySetOracle(p) then Points := [op(Points), p]; end if;
   end do;
   return Points;
  end proc:
```

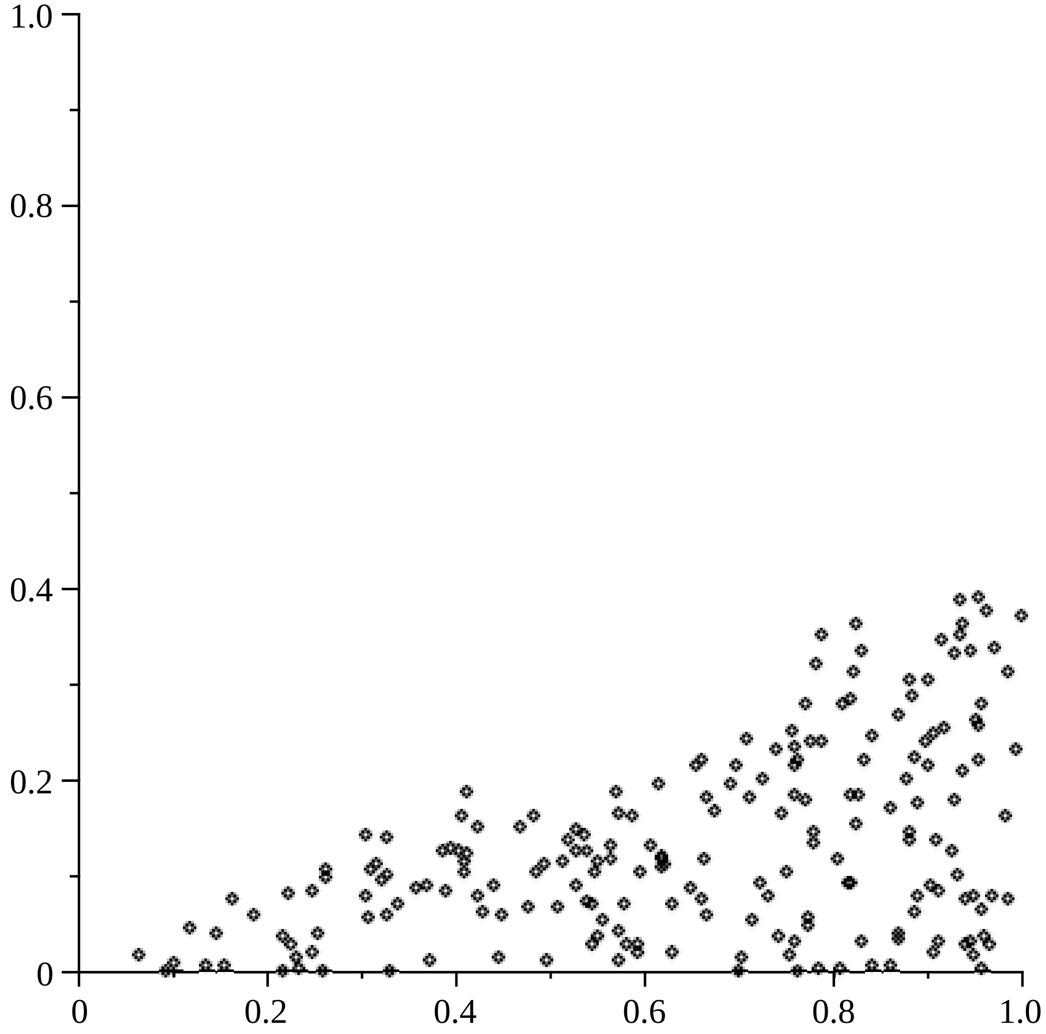
> *S* := *RandomPointsInSet*(*MySetOracle*, 50)

$$S := \left[\left[\frac{787}{1000}, \frac{171}{500}\right], \left[\frac{493}{1000}, \frac{9}{250}\right], \left[\frac{399}{500}, \frac{47}{500}\right], \left[\frac{49}{1000}, \frac{1}{250}\right], \left[\frac{23}{25}, \frac{27}{1000}\right], \left[\frac{361}{500}, \frac{49}{500}\right], \left[\frac{243}{250}, \frac{371}{1000}\right], \left[\frac{43}{250}, \frac{9}{200}\right], \left[\frac{779}{1000}, \frac{43}{500}\right], \left[\frac{999}{1000}, \frac{373}{1000}\right], \left[\frac{821}{1000}, \frac{267}{1000}\right], \left[\frac{697}{1000}, \frac{19}{125}\right], \left[\frac{599}{1000}, \frac{9}{125}\right], \left[\frac{379}{1000}, \frac{3}{20}\right]\right] \quad (3.6)$$

> *pointplot*(*S*, *view* = [0 ..1, 0 ..1])



> *pointplot*(*RandomPointsInSet*(*MySetOracle*, 1000), *view* = [0 ..1, 0 ..1])



> *pointplot*(*RandomPointsInSet*(*MySetOracle*, 10000), *view* = [0 ..1, 0 ..1])