Sequences, Limits, and Convergence

Simple Computations of Limits

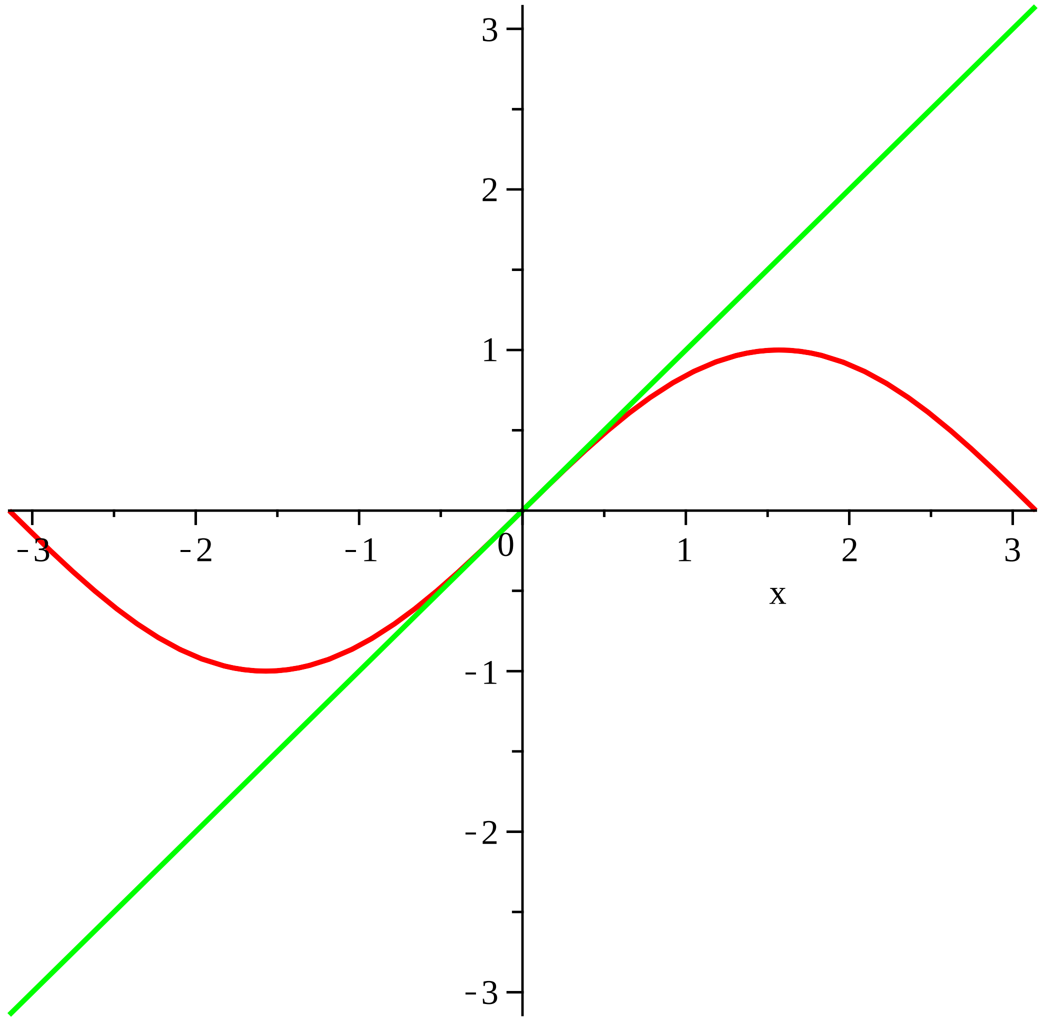
> $limit(\sin(x), x=0)$

$$0 \qquad (1.1.1)$$

> $limit(\sin(x)/x, x=0);$

$$1 \qquad (1.1.2)$$

> $plot([\sin(x), x], x=evalf(-\pi)..evalf(\pi), thickness=2)$



> $limit\left(\frac{(\text{abs}(h)-\text{abs}(0))}{h}, h=0\right)$

$$undefined \qquad (1.1.3)$$

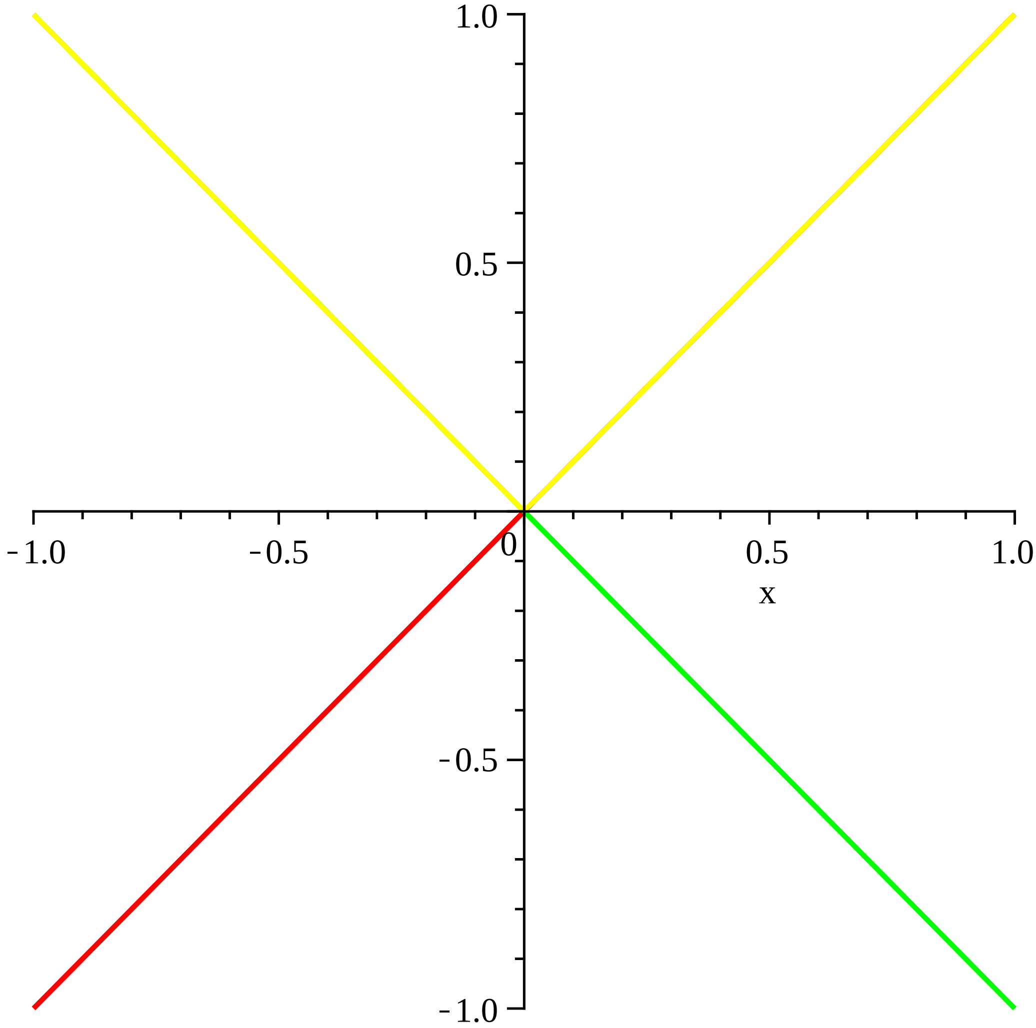
> $limit\left(\frac{(\text{abs}(h)-\text{abs}(0))}{h}, h=0, left\right)$

$$-1 \qquad (1.1.4)$$

> $limit\left(\frac{(\text{abs}(h)-\text{abs}(0))}{h}, h=0, right\right)$

$$1 \qquad (1.1.5)$$

> $plot([x,-x, \text{abs}(x)], x=-1..1, thickness=2)$



> $limit(\exp(x), x = \infty);$

$$\infty \qquad (1.1.6)$$

> $Limit(\exp(x), x = -\infty);$

$$\lim_{x \to -\infty} e^{x} \qquad (1.1.7)$$

> $value(\%)$

$$0 \qquad (1.1.8)$$

> $limit\left(\frac{1}{n}, n = \infty\right)$

$$0 \qquad (1.1.9)$$

> $limit(\sin(x), x = \infty)$

$$-1..1 \qquad (1.1.10)$$

> $with(student):$

> $expand\left(Limit\left(x^3 + 3 \cdot x^2 + x + 1, x = 4\right)\right);$

$$\left(\lim_{x \to 4} x\right)^3 + 3 \left(\lim_{x \to 4} x\right)^2 + \lim_{x \to 4} x + 1 \qquad (1.1.11)$$

> $value(\%)$

$$117 \qquad (1.1.12)$$

▼ Growth of Functions

> $limit\left(\frac{n^2}{n^3+1}, n=\infty\right)$

$$0 \quad (1.2.1)$$

> $limit\left(\frac{\pi \cdot n^3 + 17 \cdot n^2 + n}{n^3+39}, n=\infty\right)$

$$\pi \quad (1.2.2)$$

> $limit\left(\frac{n^2}{\mathrm{factorial}(n)}, n=\infty\right)$

$$0 \quad (1.2.3)$$

> $limit\left(\frac{n^k}{\mathrm{factorial}(n)}, n=\infty\right)$

$$0 \quad (1.2.4)$$

> $limit\left(\frac{n^k}{\mathrm{factorial}(n)}, n=0\right)$

$$\lim_{n \to 0} \left(\frac{n^k}{n!}\right) \quad (1.2.5)$$

> $limit\left(\frac{n^k}{\mathrm{factorial}(n)}, n=0\right)$ assuming $k > 0$

$$0 \quad (1.2.6)$$

> $limit\left(\frac{n^k}{\mathrm{factorial}(n)}, n=0\right)$ assuming $k < 0$

$$undefined \quad (1.2.7)$$

> $limit\left(\frac{n^0}{\mathrm{factorial}(n)}, n=0\right)$

$$1 \quad (1.2.8)$$

Series

> $sum(a[k], k=0..\infty)$

$$\sum_{k=0}^{\infty} a_k \quad (1.3.1)$$

> $Limit(sum(a[i], i=0..k), k=\infty)$

$$\lim_{k \to \infty} \left(\sum_{i=0}^{k} a_i\right) \quad (1.3.2)$$

> $value(\%)$

```
Error, (in match/Red1_notinx) too many levels of
recursion
```

> $sum\left(\frac{z^n}{n!}, n=0..\infty\right)$

$$e^z \quad (1.3.3)$$

> $Sum\left(\frac{(-1)^n z^{2 \cdot n+1}}{\mathrm{factorial}(2 \cdot n+1)}, n=0..\infty\right)$

$$\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \quad (1.3.4)$$

> value(%)

$$\sin(z) \quad (1.3.5)$$

> $\text{sum}\left(\frac{(-1)^n}{n}, n = 1..\infty\right)$

$$-\ln(2) \quad (1.3.6)$$

> simplify(hypergeom([1], [ ], z))

$$-\frac{1}{-1+z} \quad (1.3.7)$$

> simplify(z*hypergeom([ ], [3/2],-z^2/4));

$$\sin(z) \quad (1.3.8)$$

Special Series and Further Computations

Harmonic Series

> $sum\left(\frac{1}{n}, n = 1..\infty\right)$

$$\infty \quad (1.4.1.1)$$

Alternating Harmonic Series

> $AlternatingHarmonic := (n) \rightarrow \frac{(-1)^n}{n}$

$$AlternatingHarmonic := n \rightarrow \frac{(-1)^n}{n} \quad (1.4.2.1)$$

> map(AlternatingHarmonic, [seq(1..10)])

$$\left[-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, -\frac{1}{7}, \frac{1}{8}, -\frac{1}{9}, \frac{1}{10}\right] \quad (1.4.2.2)$$

> Sum(AlternatingHarmonic(n), n = 1..∞)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad (1.4.2.3)$$

> value(%)

$$-\ln(2) \quad (1.4.2.4)$$

Geometric Series

> $Sum(b^k, k = 0..\infty)$

$$\sum_{k=0}^{\infty} b^k \quad (1.4.3.1)$$

> value(%)

$$-\frac{1}{b-1} \quad (1.4.3.2)$$

> $sum(0^k, k = 0..\infty)$

$$0 \quad (1.4.3.3)$$

> $sum(b^k, k = 0..\infty)$ assuming $b > 1$

$$\infty \quad (1.4.3.4)$$

Computing with Series

> $\text{sum}(b^k, k = 0..12) + \text{sum}(b^k, k = 15..\infty)$

$$1 + b + b^2 + b^3 + b^4 + b^5 + b^6 + b^7 + b^8 + b^9 + b^{10} + b^{11} + b^{12} - \frac{b^{15}}{b-1} \quad (1.4.4.1)$$

> simplify(%)

$$-\frac{1-b^{13}+b^{15}}{b-1} \quad (1.4.4.2)$$

The Riemann Zeta Function

> MyZeta := $(z) \to sum\left(\frac{1}{k^z}, k=1..\infty\right)$

$$MyZeta := z \to \sum_{k=1}^{\infty} \frac{1}{k^z} \quad (1.4.5.1)$$

> MyZeta(3)

$$\zeta(3) \quad (1.4.5.2)$$

> $seq(\zeta(z), z=2..10)$

$$\frac{1}{6}\pi^2, \zeta(3), \frac{1}{90}\pi^4, \zeta(5), \frac{1}{945}\pi^6, \zeta(7), \frac{1}{9450}\pi^8, \zeta(9), \frac{1}{93555}\pi^{10} \quad (1.4.5.3)$$

> $seq(\zeta(-2\cdot k), k=1..10)$

$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \quad (1.4.5.4)$$

Riemann Hypothesis
or Riemann zeta hypothesis, n. the conjecture that the zeta function has no nontrivial zeros except on the line with re(z) = 1/2. The trivial zeros occur at negative even integers. It is known to be true for the first many million zeros, and its establishment would have many consequences for the prime number theorem and related theory.

The Riemann Series Theorem

Theorem: Let f : $\mathbb{N} \to \mathbb{R}$ be a sequence such that the series $\sum_{k=1}^{\infty} f(k)$ converges but not absolutely. For each real x there is a bijection $\beta : \mathbb{N} \to \mathbb{N}$ such that $\sum_{k=1}^{\infty} f(\beta(k)) = x$.

We want to construct such a re-ordering for given f and x. First we need two short functions which will be helpful.

```
> FindNextPositiveStartingAt := proc(f, k)
  # f as in the theorem above
  # k natural number
  # return: first index i ≥ k such that f(i) > 0
   local i; i := k;
   while f(i) ≤ 0 do i := i + 1; end do;
   return i;
  end proc:
> FindNextNegativeStartingAt := proc(f, k)
  # f as in the theorem above
  # k natural number
  # return: first index i ≥ k such that f(i) > 0
   local i; i := k;
   while f(i) ≥ 0 do i := i + 1; end do;
   return i;
  end proc:
```

The next function computes the re-ordered sequence.

```
> Riemann := proc(f, x, k)
```

```
# f as in the theorem above
# x real number
# k natural number
# return: k-th coefficient and k-th partial sum of reordered sequence
 local s, i, j, p, n;
 s := 0; i := 0;
 for j from 1 to k do
  i := i + 1;
  if x > s then
   i := FindNextPositiveStartingAt(f, i);
  else
   i := FindNextNegativeStartingAt(f, i);
  end if;
  s := s + f(i);
 end do;
 return [f(i), s];
end proc:
```

> sum(AlternatingHarmonic(n), n = 1 .. ∞)

$$-\ln(2) \quad (2.1)$$

> sum(abs(AlternatingHarmonic(n)), n = 1 .. ∞)

$$\infty \quad (2.2)$$

> seq(AlternatingHarmonic(n), n = 1 ..10)

$$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, -\frac{1}{7}, \frac{1}{8}, -\frac{1}{9}, \frac{1}{10} \quad (2.3)$$

> FindNextPositiveStartingAt(AlternatingHarmonic, 3)

$$4 \quad (2.4)$$

> Riemann(AlternatingHarmonic, 2, 1)

$$\left[\frac{1}{2}, \frac{1}{2}\right] \quad (2.5)$$

> Riemann(AlternatingHarmonic, 2, 1)[1]

$$\frac{1}{2} \quad (2.6)$$

> Riemann(AlternatingHarmonic, 2, 2)[1]

$$\frac{1}{4} \quad (2.7)$$

> Riemann(AlternatingHarmonic, 2, 2)[2]

$$\frac{3}{4} \quad (2.8)$$

> seq(Riemann(AlternatingHarmonic, 2, i)[1], i = 1 ..100)

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}, \frac{1}{18}, \frac{1}{20}, \frac{1}{22}, \frac{1}{24}, \frac{1}{26}, \frac{1}{28}, \frac{1}{30}, \frac{1}{32}, \frac{1}{34}, \quad (2.9)$$
$$\frac{1}{36}, \frac{1}{38}, \frac{1}{40}, \frac{1}{42}, \frac{1}{44}, \frac{1}{46}, \frac{1}{48}, \frac{1}{50}, \frac{1}{52}, \frac{1}{54}, \frac{1}{56}, \frac{1}{58}, \frac{1}{60}, \frac{1}{62}, -\frac{1}{63},$$
$$\frac{1}{64}, -\frac{1}{65}, \frac{1}{66}, -\frac{1}{67}, \frac{1}{68}, -\frac{1}{69}, \frac{1}{70}, -\frac{1}{71}, \frac{1}{72}, -\frac{1}{73}, \frac{1}{74}, -\frac{1}{75}, \frac{1}{76},$$
$$-\frac{1}{77}, \frac{1}{78}, -\frac{1}{79}, \frac{1}{80}, -\frac{1}{81}, \frac{1}{82}, -\frac{1}{83}, \frac{1}{84}, -\frac{1}{85}, \frac{1}{86}, -\frac{1}{87}, \frac{1}{88}, -\frac{1}{89},$$
$$-\frac{1}{91}, \frac{1}{92}, \frac{1}{94}, -\frac{1}{95}, -\frac{1}{97}, \frac{1}{98}, \frac{1}{100}, -\frac{1}{101}, -\frac{1}{103}, \frac{1}{104}, \frac{1}{106}, -\frac{1}{107},$$

$$-\frac{1}{109}, \frac{1}{110}, \frac{1}{112}, -\frac{1}{113}, -\frac{1}{115}, \frac{1}{116}, \frac{1}{118}, -\frac{1}{119}, -\frac{1}{121}, \frac{1}{122}, \frac{1}{124},$$
$$-\frac{1}{125}, -\frac{1}{127}, \frac{1}{128}, \frac{1}{130}, -\frac{1}{131}, -\frac{1}{133}, \frac{1}{134}, \frac{1}{136}, -\frac{1}{137}, -\frac{1}{139}, \frac{1}{140},$$
$$\frac{1}{142}, -\frac{1}{143}, \frac{1}{144}, -\frac{1}{145}, -\frac{1}{147}, \frac{1}{148}, -\frac{1}{149}, \frac{1}{150}$$

```
> seq(Riemann(AlternatingHarmonic, 2, i)[2], i = 1..100)
```

$$\frac{1}{2}, \frac{3}{4}, \frac{11}{12}, \frac{25}{24}, \frac{137}{120}, \frac{49}{40}, \frac{363}{280}, \frac{761}{560}, \frac{7129}{5040}, \frac{7381}{5040}, \frac{83711}{55440}, \frac{86021}{55440}, \tag{2.10}$$
$$\frac{1145993}{720720}, \frac{1171733}{720720}, \frac{1195757}{720720}, \frac{2436559}{1441440}, \frac{42142223}{24504480}, \frac{14274301}{8168160},$$
$$\frac{275295799}{155195040}, \frac{55835135}{31039008}, \frac{18858053}{10346336}, \frac{19093197}{10346336}, \frac{444316699}{237965728}, \frac{1347822955}{713897184},$$
$$\frac{34052522467}{17847429600}, \frac{34395742267}{17847429600}, \frac{312536252003}{160626866400}, \frac{315404588903}{160626866400},$$
$$\frac{9227046511387}{4658179125600}, \frac{9304682830147}{4658179125600}, \frac{290774257297357}{144403552893600}, \frac{288482137410157}{144403552893600},$$
$$\frac{581476885848239}{288807105787200}, \frac{577033699605359}{288807105787200}, \frac{52855414985869}{26255191435200},$$
$$\frac{3515057612618023}{1759097826158400}, \frac{3540926698296823}{1759097826158400}, \frac{3515432526903223}{1759097826158400},$$
$$\frac{505794642264049}{251299689451200}, \frac{35660119911296279}{17842277951035200}, \frac{35907929327282879}{17842277951035200},$$
$$\frac{2603436562940614967}{1302486290425569600}, \frac{9697839597401553857 9}{4819199274574607520 0}$$

$$\frac{260343656294061 4967}{1302486290425569600}, \frac{96978395974015538579}{48191992745746075200}, \frac{96335836070738924243}{48191992745746075200},$$
$$\frac{96969941238446109443}{48191992745746075200}, \frac{741849348261460435191 1}{371078344142244779040 0},$$
$$\frac{746606762929950752871 1}{371078344142244779040 0}, \frac{586108559273238646977 69}{293151891872373375441600},$$
$$\frac{589772957921643314170789}{293151891872373375441600}, \frac{1758461396288175373051567}{879455675617120126324800},$$
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$$\frac{602011228397938673300863650 1}{299278766412505978988329440 0}, \frac{598490301734262132371589186 1}{299278766412505978988329440 0},$$
$$\frac{25884722357779524681472499722 3}{12868986955737757096498165920 0}, \frac{25736802967483688393007003562 3}{12868986955737757096498165920 0},$$
$$\frac{25883041455617071996376300902 3}{12868986955737757096498165920 0}, \frac{229072170259418165058099261438 47}{114533983906066038158833676688 00},$$
$$\frac{2278135550516591976058043858704 7}{1145339839060660381588336766880 0},$$
$$\frac{2290584896593338284553569258344 7}{1145339839060660381588336766880 0},$$
$$\frac{10823016005941722956481192352564 09}{5383097243585103793465182804336 00},$$
$$\frac{10766351824430300811286822007255 29}{5383097243585103793465182804336 00},$$

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$$\frac{14693039224252293482129409299363 5187}{7310246056788570951525718248288 2880},$$

$$\frac{14766867155926930707435446209874271007}{738334851735645666104097543077117 0880},$$

$$\frac{151360396854311740620480998418627874 2841}{76048489728771503608722046936943060 0640},$$

$$\frac{152091632332473005078257171946867711 4001}{76048489728771503608722046936943060 0640},$$

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$$\frac{1888674251991869314786204212241086657857 70539}{940170747849061769383836747431100924098 01792},$$

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$$\frac{6268148836777920386161364116173857190637 710143437}{3134059187954847408241019797561574930480 742736320},$$

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$$\frac{520730695618102448387036565327047292769176014855920625053585007}{261241861692864143697044434225207038442983313840551693366096000},$$

$$\frac{522472308029388209345016861555215339692129236948190969676025647}{261241861692864143697044434225207038442983313840551693366096000}$$

```
> seq(evalf(Riemann(AlternatingHarmonic, 2, i)[2]), i = 1..100)
```

0.5000000000, 0.7500000000, 0.9166666667, 1.041666667, 1.141666667, **(2.11)**
1.225000000, 1.296428571, 1.358928571, 1.414484127, 1.464484127,
1.509938672, 1.551605339, 1.590066878, 1.625781163, 1.659114497,
1.690364497, 1.719776261, 1.747554039, 1.773869829, 1.798869829,
1.822679352, 1.845406625, 1.867145756, 1.887979089, 1.907979089,
1.927209858, 1.945728377, 1.963585519, 1.980826899, 1.997493565,
2.013622598, 1.997749582, 2.013374582, 1.997989966, 2.013141482,
1.998216108, 2.012921991, 1.998429237, 2.012714951, 1.998630444,
2.012519333, 1.998820703, 2.012334217, 1.999000883, 2.012158778,
1.999171765, 2.011992278, 1.999334050, 2.011834050, 1.999488371,
2.011683493, 1.999635300, 2.011540062, 1.999775356, 2.011403263,
1.999909010, 2.011272647, 2.000036692, 1.989047681, 1.999917246,
2.010555544, 2.000029228, 1.989719950, 1.999924031, 2.009924031,
2.000023041, 1.990314303, 1.999929688, 2.009363650, 2.000017856,
1.990843544, 1.999934453, 2.008863024, 2.000013467, 1.991317815,
1.999938504, 2.008413081, 2.000009719, 1.991745256, 1.999941978,

2.008006494, 2.000006494, 1.992132478, 1.999944978, 2.007637286,
2.000003698, 1.992484901, 1.999947588, 2.007300529, 2.000001259,
1.992807014, 1.999949871, 2.006992125, 1.999999118, 2.006943562,
2.000047011, 1.993244289, 2.000001046, 1.993289637, 1.999956303

> seq(evalf(Riemann(AlternatingHarmonic, 4, i)[2]), i = 1 ..100)

0.5000000000, 0.7500000000, 0.9166666667, 1.041666667, 1.141666667, **(2.12)**
1.225000000, 1.296428571, 1.358928571, 1.414484127, 1.464484127,
1.509938672, 1.551605339, 1.590066878, 1.625781163, 1.659114497,
1.690364497, 1.719776261, 1.747554039, 1.773869829, 1.798869829,
1.822679352, 1.845406625, 1.867145756, 1.887979089, 1.907979089,
1.927209858, 1.945728377, 1.963585519, 1.980826899, 1.997493565,
2.013622598, 2.029247598, 2.044399113, 2.059104995, 2.073390710,
2.087279598, 2.100793112, 2.113951007, 2.126771519, 2.139271519,
2.151466641, 2.163371403, 2.174999310, 2.186362947, 2.197474058,
2.208343623, 2.218981921, 2.229398588, 2.239602669, 2.249602669,
2.259406591, 2.269021975, 2.278455938, 2.287715197, 2.296806106,
2.305734677, 2.314506607, 2.323127297, 2.331601873, 2.339935206,
2.348131928, 2.356196444, 2.364132952, 2.371945452, 2.379637760,
2.387213517, 2.394676204, 2.402029145, 2.409275522, 2.416418379,
2.423460632, 2.430405077, 2.437254392, 2.444011149, 2.450677815,
2.457256763, 2.463750269, 2.470160526, 2.476489639, 2.482739639,
2.488912479, 2.495010040, 2.501034136, 2.506986517, 2.512868870,
2.518682824, 2.524429950, 2.530111768, 2.535729746, 2.541285301,
2.546779807, 2.552214590, 2.557590934, 2.562910083, 2.568173240,
2.573381574, 2.578536213, 2.583638254, 2.588688759, 2.593688759

> seq(evalf(Riemann(AlternatingHarmonic, 4, i)[2]), i = 1000 ..1100)

3.742735430, 3.743234931, 3.743733933, 3.744232437, 3.744730445, 3.745227958, **(2.13)**
3.745724976, 3.746221500, 3.746717532, 3.747213072, 3.747708121,
3.748202681, 3.748696752, 3.749190336, 3.749683432, 3.750176043,
3.750668169, 3.751159811, 3.751650970, 3.752141648, 3.752631844,
3.753121560, 3.753610796, 3.754099555, 3.754587836, 3.755075641,
3.755562970, 3.756049825, 3.756536207, 3.757022115, 3.757507552,
3.757992518, 3.758477014, 3.758961042, 3.759444601, 3.759927692,
3.760410318, 3.760892478, 3.761374173, 3.761855405, 3.762336175,
3.762816482, 3.763296328, 3.763775715, 3.764254642, 3.764733111,
3.765211122, 3.765688677, 3.766165777, 3.766642421, 3.767118611,
3.767594349, 3.768069634, 3.768544468, 3.769018851, 3.769492785,
3.769966270, 3.770439307, 3.770911896, 3.771384040, 3.771855738,
3.772326992, 3.772797801, 3.773268168, 3.773738093, 3.774207577,
3.774676620, 3.775145223, 3.775613388, 3.776081115, 3.776548405,
3.777015258, 3.777481676, 3.777947659, 3.778413209, 3.778878325,
3.779343009, 3.779807261, 3.780271083, 3.780734475, 3.781197438,
3.781659973, 3.782122080, 3.782583761, 3.783045015, 3.783505845,
3.783966250, 3.784426232, 3.784885790, 3.785344927, 3.785803643,
3.786261938, 3.786719813, 3.787177270, 3.787634308, 3.788090929,
3.788547134, 3.789002922, 3.789458296, 3.789913255, 3.790367800

> $\mathrm{seq}\left(\mathrm{Riemann}\left(\mathrm{AlternatingHarmonic},\ \mathrm{evalf}\left(-\frac{\pi}{2}\right), i\right)[1],\ i = 1..100\right)$

$$-1, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, \frac{1}{8}, -\frac{1}{9}, \frac{1}{10}, -\frac{1}{11}, \frac{1}{12}, -\frac{1}{13}, \frac{1}{14}, \frac{1}{16}, -\frac{1}{17}, \frac{1}{18}, -\frac{1}{19}, \tag{2.14}$$
$$-\frac{1}{21}, \frac{1}{22}, \frac{1}{24}, -\frac{1}{25}, -\frac{1}{27}, \frac{1}{28}, -\frac{1}{29}, \frac{1}{30}, \frac{1}{32}, -\frac{1}{33}, -\frac{1}{35}, \frac{1}{36}, \frac{1}{38},$$
$$-\frac{1}{39}, \frac{1}{40}, -\frac{1}{41}, -\frac{1}{43}, \frac{1}{44}, \frac{1}{46}, -\frac{1}{47}, -\frac{1}{49}, \frac{1}{50}, -\frac{1}{51}, \frac{1}{52}, \frac{1}{54}, -\frac{1}{55},$$
$$\frac{1}{56}, -\frac{1}{57}, -\frac{1}{59}, \frac{1}{60}, -\frac{1}{61}, \frac{1}{62}, \frac{1}{64}, -\frac{1}{65}, \frac{1}{66}, -\frac{1}{67}, -\frac{1}{69}, \frac{1}{70}, -\frac{1}{71},$$
$$\frac{1}{72}, \frac{1}{74}, -\frac{1}{75}, \frac{1}{76}, -\frac{1}{77}, -\frac{1}{79}, \frac{1}{80}, -\frac{1}{81}, \frac{1}{82}, \frac{1}{84}, -\frac{1}{85}, \frac{1}{86}, -\frac{1}{87},$$
$$-\frac{1}{89}, \frac{1}{90}, \frac{1}{92}, -\frac{1}{93}, -\frac{1}{95}, \frac{1}{96}, -\frac{1}{97}, \frac{1}{98}, \frac{1}{100}, -\frac{1}{101}, -\frac{1}{103}, \frac{1}{104},$$
$$\frac{1}{106}, -\frac{1}{107}, \frac{1}{108}, -\frac{1}{109}, -\frac{1}{111}, \frac{1}{112}, \frac{1}{114}, -\frac{1}{115}, -\frac{1}{117}, \frac{1}{118}, -\frac{1}{119},$$
$$\frac{1}{120}, \frac{1}{122}, -\frac{1}{123}, -\frac{1}{125}, \frac{1}{126}, \frac{1}{128}, -\frac{1}{129}, \frac{1}{130}, -\frac{1}{131}, -\frac{1}{133}$$

> $\mathrm{seq}\left(\mathrm{evalf}\left(\mathrm{Riemann}\left(\mathrm{AlternatingHarmonic},\ \mathrm{evalf}\left(-\frac{\pi}{2}\right), i\right)[2]\right),\ i = 1..100\right)$

−1., −1.333333333, −1.533333333, −1.676190476, −1.551190476, **(2.15)**
−1.662301587, −1.562301587, −1.653210678, −1.569877345, −1.646800422,
−1.575371850, −1.512871850, −1.571695380, −1.516139824, −1.568771403,
−1.616390451, −1.570935905, −1.529269239, −1.569269239, −1.606306276,
−1.570591990, −1.605074749, −1.571741415, −1.540491415, −1.570794446,
−1.599365874, −1.571588096, −1.545272307, −1.570913333, −1.545913333,
−1.570303576, −1.593559390, −1.570832118, −1.549092987, −1.570369583,
−1.590777746, −1.570777746, −1.590385589, −1.571154820, −1.552636302,
−1.570818120, −1.552960977, −1.570504837, −1.587453989, −1.570787322,
−1.587180765, −1.571051733, −1.555426733, −1.570811348, −1.555659833,
−1.570585206, −1.585077960, −1.570792246, −1.584876753, −1.570987864,
−1.557474350, −1.570807684, −1.557649789, −1.570636802, −1.583295030,
−1.570795030, −1.583140709, −1.570945587, −1.559040825, −1.570805531,
−1.559177624, −1.570671877, −1.581907832, −1.570796721, −1.559927155,
−1.570679843, −1.581206159, −1.570789493, −1.581098771, −1.570894689,
−1.560894689, −1.570795679, −1.580504417, −1.570889033, −1.561455070,
−1.570800865, −1.561541606, −1.570715917, −1.579724926, −1.570796355,
−1.562024425, −1.570720077, −1.579267086, −1.570792510, −1.579195871,
−1.570862538, −1.562665816, −1.570795898, −1.578795898, −1.570859390,
−1.563046890, −1.570798828, −1.563106520, −1.570740108, −1.578258905

>