Exercise 4: Matrices and Determinants

1. Consider the following matrices.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -4 & 3 \\ 4 & 5 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 7 & 4 \end{bmatrix}$$

Compute

(a) A^{-1}

(b) $A\,A^t$

(c) $B^t\,A\,B$

(d) $\left(2\,A + B\,B^t\right)\,A^t$

2.

(a) Compute $\det \begin{bmatrix} x^2+1 & x & 0 & 0 \\ x & x^2+1 & x & 0 \\ 0 & x & x^2+1 & x \\ 0 & 0 & x & x^2+1 \end{bmatrix}$.

(b) Compute $\det \begin{bmatrix} x^2+1 & x & 0 & 0 & 0 \\ x & x^2+1 & x & 0 & 0 \\ 0 & x & x^2+1 & x & 0 \\ 0 & 0 & x & x^2+1 & x \\ 0 & 0 & 0 & x & x^2+1 \end{bmatrix}$.

(c) Looking at the results of (a) and (b), do you have any idea what the determinant of a general matrix of the above form is? If so, check your conjecture for a 8 x 8 matrix. If not, compute the determinants for matrices of dimension 6 and 7 to get an idea.

3. Let a sphere S and a line P be given.S has its center at (0,0,0) and a radius r = 1. P intersects the sphere at two points and goes through the point (0,0,0).
Produce a drawing of S, the line P intersecting the sphere, and a tangent plane such that line and plane are orthogonal to each other.
helpful Maple-commands: with(geom3d):, sphere, intersection, TangentPlane, line, draw, detail (especially look at the intersection-object of line and sphere: *g_intersect1_s*).

4. For each natural number n, the $n \times n$ matrix A_n is defined as

$$A_n(i, j) = gcd(i, j).$$

(a) Compute the determinant of A_n for $n = 1, 2, \ldots, 15$.

(b) Try to find a closed formula for the general case.