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15th Tutorial Sheet Linear Algebra I for MCS
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(T15.1) Cramer's Rule

Consider the following matrix:

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 2 & 4 & 3 \\ 0 & 3 & 4 \end{pmatrix} \in \mathbb{R}^{(3,3)}.$$

(i) Check that $|A| = 1$.

(ii) Let $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{Z}^3$. How many solutions does $A\mathbf{x} = \mathbf{b}$ have with $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$?

(iii) Let $\mathbf{b} = (2, 1, 6) \in \mathbb{R}^3$ and solve $A\mathbf{x} = \mathbf{b}$ with $\mathbf{x} \in \mathbb{R}^3$ using Cramer's Rule.

(T15.2) Lattices

A vector $\mathbf{x}$ is called an integer linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, if $\mathbf{x}$ can be written as

$$\mathbf{x} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \ldots + \lambda_n \mathbf{v}_n,$$

with $\lambda_i \in \mathbb{Z}$. We write

$$\mathrm{span}_{\mathbb{Z}}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$$

for the set of integer linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, or the *lattice* spanned by the $\mathbf{v}_i$.

Let $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ be a labelled basis of $\mathbb{R}^n$ and $L := \mathrm{span}_{\mathbb{Z}}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ the lattice spanned by the $\mathbf{v}_i$.
Let $\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n \in L$ and let $L' := \mathrm{span}_{\mathbb{Z}}(\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n) \subseteq L$ be the sub-lattice spanned by the $\mathbf{v}'_i$.
Show that the following are equivalent:

(i) $L = L'$.

(ii) $\det(\varphi) = \pm 1$, where φ is the linear map determined by $\varphi(\mathbf{v}_i) = \mathbf{v}'_i$ for $i = 1, \ldots, n$.

(T15.3) A determinant

Let

$$A = \begin{pmatrix} a & 1 & \dots & & 1 \\ 1 & a & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \dots & 1 & a & 1 \\ 1 & \dots & & 1 & a \end{pmatrix} \in \mathbb{F}^{(n,n)}.$$

Show by induction that $\det(A) = (a-1)^{n-1}(a-1+n)$.