

January 30, 2007

14th Tutorial Sheet Linear Algebra I for MCS Winter Term 2006/2007

(T14.1) Determinants and inverses

(i) What does it mean for the determinant of a matrix $A \in \mathbb{F}_2^{(n,n)}$ to be 1?

(ii) Compute the determinant of $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \in \mathbb{F}_2^{(3,3)}$.

(iii) Compute the inverse of $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \in \mathbb{F}_2^{(3,3)}$.

(T14.2) Matrix groups

Recall that the group $\text{GL}_n(\mathbb{R})$ consists of the regular $n \times n$ matrices over \mathbb{R} , with matrix multiplication as group operation (Definition 3.3.12 in the notes). Recall also (Definition 4.1.13) that for $A = (a_{ij}) \in \mathbb{F}^{(n,n)}$ the entries of the transpose A^T are given as $(a_{ij}^T) = a_{ji}$.

We consider the following subsets (in fact subgroups):

$$G := \{A \in \text{GL}_n(\mathbb{R}) : \det(A) = \pm 1\},$$

$$H := \{A \in \text{GL}_n(\mathbb{R}) : A^T A = E_n\} (= \text{O}_n(\mathbb{R})),$$

$$K := \{A \in H : a_{ij} \in \mathbb{N} \text{ for } 1 \leq i, j \leq n\}.$$

- (i) Show that G is a subgroup of $\text{GL}_n(\mathbb{R})$.
- (ii) Verify that for arbitrary matrices $A, B \in \mathbb{F}^{(n,n)}$: $(AB)^T = B^T A^T$ and $(A^T)^T = A$. Use this to show that H is a subgroup of $\text{GL}_n(\mathbb{R})$. Why is H also a subgroup of G ? (H is called the *orthogonal group* over \mathbb{R} and usually denoted by $\text{O}_n(\mathbb{R})$.)
- (iii) Show that K is a subgroup of H .
- (iv) Give a concrete description of the elements of K . (Hint: show that the only possible entries for a matrix $A \in K$ are 0 and 1.) Then use this to prove that K is isomorphic to the permutation group S_n .