Fachbereich Mathematik
Martin Otto
Benno van den Berg, Sven Herrmann

TECHNISCHE UNIVERSITÄT DARMSTADT

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14th Tutorial Sheet Linear Algebra I for MCS
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(T14.1) Determinants and inverses

(i) What does it mean for the determinant of a matrix $A \in \mathbb{F}_2^{(n,n)}$ to be 1?

(ii) Compute the determinant of $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \in \mathbb{F}_2^{(3,3)}$.

(iii) Compute the inverse of $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \in \mathbb{F}_2^{(3,3)}$.

(T14.2) Matrix groups

Recall that the group $\mathrm{GL}_n(\mathbb{R})$ is consists of the regular $n \times n$ matrices over $\mathbb{R}$, with matrix multiplication as group operation (Definition 3.3.12 in the notes). Recall also (Definition 4.1.13) that for $A = (a_{ij}) \in \mathbb{F}^{(n,n)}$ the entries of the transpose A^T are given as $(a^T_{ij}) = a_{ji}$.

We consider the following subsets (in fact subgroups):

$$\begin{aligned} G &:= \{A \in \mathrm{GL}_n(\mathbb{R}) \,:\, \det(A) = \pm 1\}, \\ H &:= \{A \in \mathrm{GL}_n(\mathbb{R}) \,:\, A^T A = E_n\}\big(= \mathrm{O}_n(\mathbb{R})\big), \\ K &:= \{A \in H \,:\, a_{ij} \in \mathbb{N} \text{ for } 1 \leqslant, i, j \leqslant n\}. \end{aligned}$$

(i) Show that G is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.

(ii) Verify that for arbitrary matrices $A, B \in \mathbb{F}^{(n,n)}$: $(AB)^T = B^T A^T$ and $(A^T)^T = A$. Use this to show that H is a subgroup of $\mathrm{GL}_n(\mathbb{R})$. Why is H also a subgroup of G? (H is called the *orthogonal group* over $\mathbb{R}$ and usually denoted by $\mathrm{O}_n(\mathbb{R})$.)

(iii) Show that K is a subgroup of H.

(iv) Give a concrete description of the elements of K. (Hint: show that the only possible entries for a matrix $A \in K$ are 0 and 1.) Then use this to prove that K is isomorphic to the permutation group S_n.