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9th Tutorial Sheet Linear Algebra I for MCS
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(T9.1) Linear maps

(i) Let $\varphi\colon V \to W$ be a linear map between $\mathbb{F}$-vector spaces. Show that if φ is bijective, then its inverse $\varphi^{-1}\colon W \to V$ is also linear.

(ii) Let $\varphi\colon V \to W$ be a linear map between $\mathbb{F}$-vector spaces. Show that the image $\varphi(U)$ of any subspace U of V is a subspace of W.

(T9.2) Projection

Let W be the 2-dimensional subspace of $\mathbb{R}^3$ spanned by $\mathbf{w}_1 = (1, 0, 2)$ and $\mathbf{w}_2 = (2, -1, 4)$, and let U be the 1-dimensional subspace spanned by $\mathbf{u} = (-1, 1, 2)$.
The map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ projects vectors onto W in the direction of U.
[In intuitive terms: think of the plane W as a screen, and the vector $\mathbf{u}$ as pointing in the direction of the sun light. Then φ maps every vector to its shadow on the screen W.]

(i) Describe φ formally in terms of the representation $\mathbb{R}^3 = U \oplus W$. Show that φ is linear and represent φ in terms of a suitable basis of $\mathbb{R}^3$.

(ii) Show that $\varphi \circ \varphi = \varphi$.

(iii) Determine the image and kernel of φ and show that $\mathbb{R}^3 = \ker(\varphi) \oplus \operatorname{image}(\varphi)$.

(T9.3) Projection, continued

A linear map $\varphi : V \to V$ is called *a projection* if $\varphi \circ \varphi = \varphi$.

(i) Show that $V = \ker(\varphi) + \operatorname{image}(\varphi)$.
Hint: For $\mathbf{v} \in V$, consider $\mathbf{u} = \mathbf{v} - \varphi(\mathbf{v})$.

(ii) Show that $\ker(\varphi) \cap \operatorname{image}(\varphi) = \{\mathbf{0}\}$.

Note that (i) and (ii) together show that V is the direct sum of $\ker(\varphi)$ and $\operatorname{image}(\varphi)$.