Fachbereich Mathematik
Martin Otto
Benno van den Berg, Sven Herrmann

TECHNISCHE UNIVERSITÄT DARMSTADT

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7th Tutorial Sheet Linear Algebra I for MCS
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(T7.1) Linear Independence Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be linearly independent in an $\mathbb{F}$-vector space V, where $1+1 \neq 0$ in $\mathbb{F}$. Show that then $\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}$ and $\mathbf{u}-2\mathbf{v}+\mathbf{w}$ are also linearly independent.

The following two exercises illustrate two useful applications of the Gauß–Jordan procedure in the search for bases of subspaces or for the analysis of linear dependencies in some $\mathbb{F}^n$.

(T7.2) Basis for a span of row vectors

(i) Let $\mathbb{F}$ be a field, and consider a matrix with entries from $\mathbb{F}$ of the following echelon form:

$$\begin{pmatrix} 0 & 0 & * & \times & \times & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & * & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & 0 & * & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Here $\times$ can be any element from $\mathbb{F}$, $*$ stands for field elements different from 0. More formally: we consider a matrix whose first non-zero entry in any row occurs (strictly) later than the first non-zero entry in the previous row. Prove that the non-zero row vectors are linearly independent.

(ii) Using (i) and Gauß–Jordan row transformations on the basis of Lemma 2.4.4, find a basis for the subspace in $\mathbb{R}^6$ spanned by the following vectors:

$$\begin{aligned} \mathbf{v}_1 &= (1, 2, 1, 4, -2, 0) \\ \mathbf{v}_2 &= (2, 6, -1, -1, 1, 3) \\ \mathbf{v}_3 &= (4, 10, 2, 12, -8, 2) \\ \mathbf{v}_4 &= (5, 9, 4, 12, 0, 1) \end{aligned}$$

(T7.3) Linear dependencies among the column vectors

Suppose a matrix A over a field $\mathbb{F}$ with *column* vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ has been transformed using the Gauß–Jordan *row transformations* (T1), (T2), (T3) into a matrix B with *column* vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$.

(i) Prove that for any sequence of elements $\lambda_i \in \mathbb{F}$ $(1 \leqslant i \leqslant n)$ the following holds:

$$\sum_{i=1}^{n} \lambda_i \mathbf{a}_i = \mathbf{0} \Leftrightarrow \sum_{i=1}^{n} \lambda_i \mathbf{b}_i = \mathbf{0}.$$

(ii) Prove that (i) implies that for every set $I \subseteq \{1, 2, \ldots, n\}$ of indices, the set $\{\mathbf{a}_i : i \in I\}$ is linearly (in)dependent if, and only if, the set $\{\mathbf{b}_i : i \in I\}$ is linearly (in)dependent.