Fachbereich Mathematik
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November 21, 2006

6th Tutorial Sheet Linear Algebra I for MCS
Winter Term 2006/2007

(T6.1) Properties of spans

(Cf. Exercise 2.4.2 on page 59 of the notes.) Let S and S' be two arbitrary subsets of an $\mathbb{F}$-vector space V.

(i) Show that $\mathrm{span}(S) \subseteq \mathrm{span}(S')$ whenever $S \subseteq S'$;
and that $\mathrm{span}(\mathrm{span}(S)) = \mathrm{span}(S)$.

(ii) Show that in general *not* $\mathrm{span}(S) \cap \mathrm{span}(S') = \mathrm{span}(S \cap S')$
and *not* $\mathrm{span}(S) \cup \mathrm{span}(S') = \mathrm{span}(S \cup S')$.
Which inclusions do hold in these cases?

(T6.2) Bases

Let V be an $\mathbb{F}$-vector space. Show that the following are equivalent:

(i) $B \subseteq V$ is a basis [cf. Definition 2.5.1].

(ii) B is linearly independent and every $B' \supsetneq B$ is linearly dependent.
[B is maximally independent].

(iii) $\mathrm{span}(B) = V$ and for every $\mathbf{b} \in B$, $\mathrm{span}(B \setminus \{\mathbf{b}\}) \subsetneq V$.
[B is minimally spanning].

(T6.3) Affine subspaces

Let $\mathbb{F}$ be a field, and V an $\mathbb{F}$-vector space.

(i) Show that for any non-empty subset $S_0 \subseteq V$ there is a unique $\subseteq$-minimal affine subspace S of V such that $S_0 \subseteq S$.
(In case $S_0 = \{\mathbf{u}, \mathbf{v}\}$ consists of two distinct points, the corresponding affine subspace is the *line* through $\mathbf{u}$ and $\mathbf{v}$.)

(ii) Assume now that $1 + 1 \neq 0$ in $\mathbb{F}$ ("$\mathbb{F}$ not of characteristic 2"). Show that then the following are equivalent for any non-empty subset $S \subseteq V$:

(1) S is an affine subspace.

(2) S is "line-closed" in the sense that for all $\mathbf{u}, \mathbf{v} \in S$ with $\mathbf{u} \neq \mathbf{v}$ the line through $\mathbf{u}, \mathbf{v}$ is contained in S.

Is the assumption that $1 + 1 \neq 0$ necessary?