

6th Tutorial Sheet Linear Algebra I for MCS Winter Term 2006/2007

(T6.1) Properties of spans

(Cf. Exercise 2.4.2 on page 59 of the notes.) Let S and S' be two arbitrary subsets of an \mathbb{F} -vector space V .

- (i) Show that $\text{span}(S) \subseteq \text{span}(S')$ whenever $S \subseteq S'$;
and that $\text{span}(\text{span}(S)) = \text{span}(S)$.
- (ii) Show that in general *not* $\text{span}(S) \cap \text{span}(S') = \text{span}(S \cap S')$
and *not* $\text{span}(S) \cup \text{span}(S') = \text{span}(S \cup S')$.
Which inclusions do hold in these cases?

(T6.2) Bases

Let V be an \mathbb{F} -vector space. Show that the following are equivalent:

- (i) $B \subseteq V$ is a basis [cf. Definition 2.5.1].
- (ii) B is linearly independent and every $B' \supsetneq B$ is linearly dependent.
[B is maximally independent].
- (iii) $\text{span}(B) = V$ and for every $\mathbf{b} \in B$, $\text{span}(B \setminus \{\mathbf{b}\}) \subsetneq V$.
[B is minimally spanning].

(T6.3) Affine subspaces

Let \mathbb{F} be a field, and V an \mathbb{F} -vector space.

- (i) Show that for any non-empty subset $S_0 \subseteq V$ there is a unique \subseteq -minimal affine subspace S of V such that $S_0 \subseteq S$.
(In case $S_0 = \{\mathbf{u}, \mathbf{v}\}$ consists of two distinct points, the corresponding affine subspace is the *line* through \mathbf{u} and \mathbf{v} .)
- (ii) Assume now that $1 + 1 \neq 0$ in \mathbb{F} (“ \mathbb{F} not of characteristic 2”). Show that then the following are equivalent for any non-empty subset $S \subseteq V$:
 - (1) S is an affine subspace.
 - (2) S is “line-closed” in the sense that for all $\mathbf{u}, \mathbf{v} \in S$ with $\mathbf{u} \neq \mathbf{v}$ the line through \mathbf{u}, \mathbf{v} is contained in S .

Is the assumption that $1 + 1 \neq 0$ necessary?