

November 14, 2006

5th Tutorial Sheet Linear Algebra I for MCS Winter Term 2006/2007

(T5.1) Complex numbers

As outlined on page 41 of the notes, the complex numbers are represented by expressions of the form

$$z = a + bi$$

with $a, b \in \mathbb{R}$, $i \notin \mathbb{R}$ a new constant. Two such expressions represent the same complex number if, and only if, they agree in a and b (the real and the imaginary part of z).

Addition and multiplication of complex numbers are defined as follows:

$$\begin{aligned}(a + bi) + (c + di) &:= (a + c) + (b + d)i, \\ (a + bi)(c + di) &:= (ac - bd) + (ad + bc)i.\end{aligned}$$

Identifying $a \in \mathbb{R}$ with the complex number $a + 0i$ and the new constant i with $0 + 1i$, one may regard these as the natural extensions of addition and multiplication in \mathbb{R} based on associativity, commutativity, distributivity and the identity $i^2 = -1$.

- (i) Show that $(\mathbb{C}, +, \cdot, 0, 1)$ is a commutative ring and that \mathbb{R} is a subring of \mathbb{C} if we identify $a \in \mathbb{R}$ with $a + 0i \in \mathbb{C}$.
- (ii) The complex numbers can be represented as vectors in the 2-dimensional real plane (the complex plane, or Gauß plane) via the bijective correspondence $\varphi: a + bi \mapsto (a, b)$ between \mathbb{C} and \mathbb{R}^2 . Give a geometric interpretation of complex addition, and the following functions in terms of this correspondence with \mathbb{R}^2 :

$$\begin{aligned}\operatorname{Re}(a + bi) &:= a && \text{(real part)} \\ \operatorname{Im}(a + bi) &:= b && \text{(imaginary part)} \\ \overline{a + bi} &:= a - bi && \text{(complex conjugate)} \\ |a + bi| &:= \sqrt{a^2 + b^2} && \text{(absolute value)} \\ \arg(a + bi) &:= \arctan\left(\frac{b}{a}\right) \text{ for } a > 0, b \geq 0 && \text{(argument)}\end{aligned}$$

Verify that $|z|^2 = z\bar{z}$ for every complex number z .

- (iii) Show that \mathbb{C} is a field, and $z \mapsto \bar{z}$ is a an automorphism of the structure $(\mathbb{C}, +, \cdot, 0, 1)$ (a field automorphism¹.)

¹For those seeking a challenge: do \mathbb{Q} or \mathbb{R} have any field automorphisms besides the identity?

- (iv) For a fixed $c \in \mathbb{C} \setminus \{0\}$, give a geometric description of the transformation that the function $z \mapsto cz$ in \mathbb{C} induces on \mathbb{R}^2 via the correspondence φ from part (ii). Consider, for instance, $c = 2$, $c = i$, $c = 4 + 3i$, $c = 4 - 3i$. What, in general, is the geometric interpretation of complex multiplication?

(T5.2) Rings of matrices

A 2-by-2 real matrix is an object of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with } a, b, c, d \in \mathbb{R}.$$

Addition and multiplication on the set $\mathbb{R}^{(2,2)}$ of 2 by 2 real matrices are defined as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

- (i) Show that $(\mathbb{R}^{(2,2)}, +, \cdot, \mathbf{0}, E_2)$, where $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, is a ring, that is not commutative.

- (ii) Show that the following are subrings of $\mathbb{R}^{(2,2)}$:

(a) $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{(2,2)} : c = 0 \right\}$

(b) $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{(2,2)} : b = c = 0 \right\}$

(c) $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{(2,2)} : a = d, b = -c \right\}$

- (iii) Check for isomorphisms between the additive groups or the full ring structures in these subrings with other familiar structures. In particular, one of these subrings is actually a field, isomorphic to \mathbb{C} .