Fachbereich Mathematik
Martin Otto
Benno van den Berg, Sven Herrmann

TECHNISCHE UNIVERSITÄT DARMSTADT

November 14, 2006

5th Tutorial Sheet Linear Algebra I for MCS
Winter Term 2006/2007

(T5.1) Complex numbers

As outlined on page 41 of the notes, the complex numbers are represented by expressions of the form

$$z = a + bi$$

with $a, b \in \mathbb{R}$, $i \notin \mathbb{R}$ a new constant. Two such expressions represent the same complex number if, and only if, they agree in a and b (the real and the imaginary part of z).
Addition and multiplication of complex numbers are defined as follows:

$$\begin{aligned} (a+bi)+(c+di) &:= (a+c)+(b+d)i, \\ (a+bi)(c+di) &:= (ac-bd)+(ad+bc)i. \end{aligned}$$

Identifying $a \in \mathbb{R}$ with the complex number $a + 0i$ and the new constant i with $0 + 1i$, one may regard these as the natural extensions of addition and multiplication in $\mathbb{R}$ based on associativity, commutativity, distributivity and the identity $i^2 = -1$.

(i) Show that $(\mathbb{C}, +, \cdot, 0, 1)$ is a commutative ring and that $\mathbb{R}$ is a subring of $\mathbb{C}$ if we identify $a \in \mathbb{R}$ with $a + 0i \in \mathbb{C}$.

(ii) The complex numbers can be represented as vectors in the 2-dimensional real plane (the complex plane, or Gauß plane) via the bijective correspondence $\varphi\colon a+bi \mapsto (a,b)$ between $\mathbb{C}$ and $\mathbb{R}^2$. Give a geometric interpretation of complex addition, and the following functions in terms of this correspondence with $\mathbb{R}^2$:

$$\begin{array}{lll} \mathrm{Re}(a+bi) & := a & \text{(real part)} \\ \mathrm{Im}(a+bi) & := b & \text{(imaginary part)} \\ \overline{a+bi} & := a-bi & \text{(complex conjugate)} \\ |a+bi| & := \sqrt{a^2+b^2} & \text{(absolute value)} \\ \arg(a+bi) & := \arctan\left(\frac{b}{a}\right) \quad \text{for } a>0, b\geqslant 0 & \text{(argument)} \end{array}$$

Verify that $|z|^2 = z\overline{z}$ for every complex number z.

(iii) Show that $\mathbb{C}$ is a field, and $z \mapsto \overline{z}$ is a an automorphism of the structure $(\mathbb{C}, +.\cdot, 0, 1)$ (a field automorphism[1].)

[1]For those seeking a challenge: do $\mathbb{Q}$ or $\mathbb{R}$ have any field automorphisms besides the identity?

(iv) For a fixed $c \in \mathbb{C} \setminus \{0\}$, give a geometric description of the transformation that the function $z \mapsto cz$ in $\mathbb{C}$ induces on $\mathbb{R}^2$ via the correspondence φ from part (ii). Consider, for instance, $c = 2$, $c = i$, $c = 4 + 3i$, $c = 4 - 3i$. What, in general, is the geometric interpretation of complex multiplication?

(T5.2) Rings of matrices

A 2-by-2 real matrix is an object of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with } a, b, c, d \in \mathbb{R}.$$

Addition and multiplication on the set $\mathbb{R}^{(2,2)}$ of 2 by 2 real matrices are defined as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

(i) Show that $(\mathbb{R}^{(2,2)}, +, \cdot, \mathbf{0}, E_2)$, where $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, is a ring, that is not commutative.

(ii) Show that the following are subrings of $\mathbb{R}^{(2,2)}$:

(a) $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{(2,2)} : c = 0\}$

(b) $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{(2,2)} : b = c = 0\}$

(c) $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{(2,2)} : a = d, b = -c\}$

(iii) Check for isomorphisms between the additive groups or the full ring structures in these subrings with other familiar structures. In particular, one of these subrings is actually a field, isomorphic to $\mathbb{C}$.