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3rd Tutorial Sheet Linear Algebra I for MCS
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(T3.1) Congruence modulo n

Let $a, b \in \mathbb{Z}$ be two integers, and $n > 0$ a natural number. Show that the following two conditions are equivalent:

(i) If you divide a by n and b by n, the remainders are the same.

(ii) $n \mid a - b \quad$ (n divides $a - b$).

(T3.2) Symmetries

Consider infinite black-and-white pixel patterns formalised as functions $p \colon \mathbb{Z}^2 \to \{0, 1\}$ (think of $p(i, j) = 1$ as the pixel in position (i, j) being white, $p(i, j) = 0$ as it being black). A pair $(k, \ell) \in \mathbb{Z}^2$ induces a bijection $\rho_{k,\ell} \in \mathrm{Sym}(\mathbb{Z}^2)$ according to

$$\begin{aligned} \rho_{k,\ell} \colon \mathbb{Z}^2 &\longrightarrow \mathbb{Z}^2 \\ (i, j) &\longmapsto (i + k, j + \ell). \end{aligned}$$

$\rho_{k,\ell}$ is a *symmetry* of the pattern p if $p(\rho_{k,\ell}(i, j)) = p(i, j)$ for all $(i, j) \in \mathbb{Z}^2$, i.e., if $p \circ \rho = p$. For a fixed pattern p let $\mathrm{sym}(p) := \{\rho_{k,\ell} \colon \rho_{k,\ell} \text{ a symmetry of } p\}$.

(i) Show that, for every pattern p, $\mathrm{sym}(p) := \{\rho_{k,\ell} \colon \rho_{k,\ell} \text{ a symmetry of } p\}$ forms a group w.r.t. composition.

(ii) Under which operations is the set of vectors $\{(k, \ell) \in \mathbb{Z}^2 \colon \rho_{k,\ell} \in \mathrm{sym}(p)\}$ closed? What is the relationship between $\mathrm{sym}(p)$ and the group formed by vector addition in the (real or rational) plane?

(iii) How could you describe the whole pattern p in terms of $\mathrm{sym}(p)$ and as little information as possible about actual pixel values $p(i, j)$? Discuss and look at some examples.

(T3.3) Group theory

Let G be a non-empty set and $* \colon G \times G \to G$ be a binary operation, $e \in G$. Let moreover the following be satisfied:

H1 for all $a, b, c \in G$: $(a * b) * c = a * (b * c)$.

H2 for all $a \in G$: $e * a = a$. [e a left neutral element]

H3 for all $a \in G$ there is a $b \in G$ such that $b * a = e$. [existence of left inverse]

Show that G is a group (compare section 1.3.2 in the lecture notes).