

2. Tutorial Sheet Linear Algebra I for MCS Winter Term 2006/2007

(T2.1) Solving a system of linear equations

Solve the following system of linear equations using Gauß-Jordan elimination:

$$\begin{array}{rccccrcr} & & 2 \cdot x_2 & + & 3 \cdot x_3 & - & 5 \cdot x_4 & = & 1 \\ 1 \cdot x_1 & + & 3 \cdot x_2 & & & & - & 4 \cdot x_4 & = & -1 \\ 1 \cdot x_1 & + & 5 \cdot x_2 & + & 1 \cdot x_3 & - & 5 \cdot x_4 & = & -4 \\ -2 \cdot x_1 & - & 2 \cdot x_2 & + & 5 \cdot x_3 & & & = & 2 \end{array}$$

Relate the solution set of the inhomogeneous system to the solution set of the homogeneous system and give a “nice” parametric representation.

(T2.2) Systems of linear equations over \mathbb{Z}_2

Consider the following system of linear equations over \mathbb{Z}_2 (the field $\mathbb{F}_2 = (\mathbb{Z}_2, +_2, \cdot_2, 0, 1)$ of two elements):

$$\begin{array}{rcccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & & & = & 0 \\ x_1 & + & x_2 & & & + & x_5 & + & x_6 & = & 0 \\ x_1 & & & + & x_3 & & + & x_5 & & + & x_7 = & 0 \end{array}$$

Solve this homogeneous system using (the \mathbb{Z}_2 analogue of) Gauß-Jordan. Determine the number of solutions.

(T2.3) Hamming code, vector spaces over \mathbb{Z}_2

Consider \mathbb{Z}_2^7 as the space of bit strings of length 7.

We define the distance d between two vectors (bit strings) \mathbf{x} and \mathbf{y} in \mathbb{Z}_2^7 according to

$$d(\mathbf{x}, \mathbf{y}) := |\{i: 1 \leq i \leq 7, x_i \neq y_i\}|.$$

(i) Show that d has the following properties:

$$\left. \begin{array}{l} d(\mathbf{x}, \mathbf{y}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{y} \\ d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x}) \quad \text{(symmetry)} \\ d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \quad \text{(triangle inequality)} \\ d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x} - \mathbf{y}, 0) \quad \text{(translation invariance)} \end{array} \right\} d \text{ is a metric}$$

Let $C \subseteq \mathbb{Z}_2^7$ be the solution set of the homogeneous system from the previous exercise, T2.2. Consider C as a set of codes. We want to exploit the properties of this subset for error correction. Let

$$d(C) := \min\{d(\mathbf{x}, \mathbf{y}) : \mathbf{x} \neq \mathbf{y} \in C\}$$

be the so-called *minimal distance* of C .

(ii) Show that $d(C) = 3$.

For this proceed as follows. First show that $d(C) = \min\{d(\mathbf{x}, \mathbf{0}) : \mathbf{0} \neq \mathbf{x} \in C\}$. Next show that the minimal distance is at most 3, then eliminate in turn the possibilities that the distance is 1 or 2.

(iii) Suppose A wants to transmit to B a certain element $\mathbf{c} \in C$. Explain that the result in (ii) shows that C is robust in the sense that the element $\mathbf{c} \in C$ can be recovered when up to one bit is corrupted in transmission (flipped from 0 to 1 or vice versa).

Suppose B receives $\mathbf{e} = (0, 1, 1, 1, 0, 0, 1)$ and assume that no more than one bit is distorted. Recover the correct \mathbf{c} .