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2. Tutorial Sheet Linear Algebra I for MCS
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(T2.1) Solving a system of linear equations

Solve the following system of linear equations using Gauß-Jordan elimination:

$$\begin{array}{rcrcrcrcr} & & 2\cdot x_2 & + & 3\cdot x_3 & - & 5\cdot x_4 & = & 1 \\ 1\cdot x_1 & + & 3\cdot x_2 & & & - & 4\cdot x_4 & = & -1 \\ 1\cdot x_1 & + & 5\cdot x_2 & + & 1\cdot x_3 & - & 5\cdot x_4 & = & -4 \\ -2\cdot x_1 & - & 2\cdot x_2 & + & 5\cdot x_3 & & & = & 2 \end{array}$$

Relate the solution set of the inhomogeneous system to the solution set of the homogeneous system and give a "nice" parametric representation.

(T2.2) Systems of linear equations over $\mathbb{Z}_2$

Consider the following system of linear equations over $\mathbb{Z}_2$ (the field $\mathbb{F}_2 = (\mathbb{Z}_2, +_2, \cdot_2, 0, 1)$ of two elements):

$$\begin{array}{ccccccccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & & & & & & & = & 0 \\ x_1 & + & x_2 & & & & & + & x_5 & + & x_6 & & & = & 0 \\ x_1 & & & + & x_3 & & & + & x_5 & & & + & x_7 & = & 0 \end{array}$$

Solve this homogeneous system using (the $\mathbb{Z}_2$ analogue of) Gauß-Jordan.
Determine the number of solutions.

(T2.3) Hamming code, vector spaces over $\mathbb{Z}_2$

Consider $\mathbb{Z}_2^7$ as the space of bit strings of length 7.
We define the distance d between two vectors (bit strings) $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{Z}_2^7$ according to

$$d(\mathbf{x},\mathbf{y}) := |\{i\colon\ 1 \leqslant i \leqslant 7, x_i \neq y_i\}|.$$

(i) Show that d has the following properties:

$$\left.\begin{array}{ll} d(\mathbf{x},\mathbf{y}) = 0 \ \Leftrightarrow\ \mathbf{x} = \mathbf{y} & \\ d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{x}) & \text{(symmetry)} \\ d(\mathbf{x},\mathbf{z}) \leqslant d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}) & \text{(triangle inequality)} \end{array}\right\}\ d \text{ is a } \textit{metric}$$

$$d(\mathbf{x},\mathbf{y}) = d(\mathbf{x}-\mathbf{y}, 0) \qquad \text{(translation invariance)}$$

Let $C \subseteq \mathbb{Z}_2^7$ be the solution set of the homogeneous system from the previous exercise, T2.2. Consider C as a set of codes. We want to exploit the properties of this subset for error correction. Let

$$d(C) := \min\{d(\mathbf{x}, \mathbf{y}) : \mathbf{x} \neq \mathbf{y} \in C\}$$

be the so-called *minimal distance* of C.

(ii) Show that $d(C) = 3$.

For this proceed as follows. First show that $d(C) = \min\{d(\mathbf{x}, \mathbf{0}) : \mathbf{0} \neq \mathbf{x} \in C\}$. Next show that the minimal distance is at most 3, then eliminate in turn the possibilities that the distance is 1 or 2.

(iii) Suppose A wants to transmit to B a certain element $\mathbf{c} \in C$. Explain that the result in (ii) shows that C is robust in the sense that the element $\mathbf{c} \in C$ can be recovered when up to one bit is corrupted in transmission (flipped from 0 to 1 or vice versa).

Suppose B receives $\mathbf{e} = (0, 1, 1, 1, 0, 0, 1)$ and assume that no more than one bit is distorted. Recover the correct $\mathbf{c}$.