

October 17, 2006

1. Tutorial Sheet Linear Algebra I for MCS Winter Term 2006/2007

(T1.1) [vector space axioms]

Verify (a few of) the vector space axioms for the vector spaces \mathbb{R}^2 as well as for \mathbb{R}^n , $n \geq 3$.

(T1.2) [some vector space properties]

Derive the following for arbitrary vectors \mathbf{v} , \mathbf{w} and scalars λ from the vector space axioms:

- (i) If $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$, then $\mathbf{u} = \mathbf{v}$. Special case: if $\mathbf{u} + \mathbf{w} = \mathbf{w}$, then $\mathbf{u} = \mathbf{0}$.
- (ii) $\lambda \cdot \mathbf{0} = \mathbf{0}$ and $0 \cdot \mathbf{v} = \mathbf{0}$.
- (iii) $-(\lambda \mathbf{v}) = (-\lambda)\mathbf{v}$.

(T1.3) [\mathbb{Z}_2^n as a vector space]

Consider $\mathbb{Z}_2 = \{0, 1\}$ with operations of addition and multiplication obtained from the ordinary integer operations through reduction modulo 2.

Provide the tables for addition and multiplication in \mathbb{Z}_2 and check out some basic arithmetical properties (cf. Exercise 1.1.5 in the lecture notes).

Is it clear how to turn $\mathbb{Z}_2^n := (\mathbb{Z}_2)^n$, the set of n -tuples over \mathbb{Z}_2 , into a vector space?

- (i) How can you visualize the space \mathbb{Z}_2^3 ?
- (ii) What do the lines and planes (through the origin $\mathbf{0} = (0, 0)$) in \mathbb{Z}_2^3 look like?
- (iii) How many lines (through the origin) are there in \mathbb{Z}_2^3 (in \mathbb{Z}_2^n)?

(T1.4) [\mathbb{Z}_2^n and subsets]

One may consider the values 0 and 1 of \mathbb{Z}_2 as the logical values *true* and *false*, or as bits. The elements of \mathbb{Z}_2^n then correspond to length n bit strings, as, e.g., $\mathbf{b} = 0100111$ for $n = 7$. A bit string $\mathbf{b} = b_1 \dots b_n$ of length n may also be regarded as an encoding of a subset $S_{\mathbf{b}}$ of the set $\{1, \dots, n\}$:

$$S_{\mathbf{b}} := \{i : b_i = 1\}.$$

- (i) In \mathbb{Z}_2 : show that addition and multiplication correspond to the logical operations XOR (exclusive or) and AND, respectively.
- (ii) In \mathbb{Z}_2^n : how can you interpret vector addition and scalar multiplication in terms of length n bit strings, and in terms of subsets of $\{1, \dots, n\}$, respectively?