Fachbereich Mathematik
Martin Otto
Benno van den Berg, Sven Herrmann

TECHNISCHE UNIVERSITÄT DARMSTADT

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1. Tutorial Sheet Linear Algebra I for MCS
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(T1.1) [vector space axioms]

Verify (a few of) the vector space axioms for the vector spaces $\mathbb{R}^2$ as well as for $\mathbb{R}^n$, $n \geqslant 3$.

(T1.2) [some vector space properties]

Derive the following for arbitrary vectors $\mathbf{v}, \mathbf{w}$ and scalars λ from the vector space axioms:

(i) If $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$, then $\mathbf{u} = \mathbf{v}$. Special case: if $\mathbf{u} + \mathbf{w} = \mathbf{w}$, then $\mathbf{u} = \mathbf{0}$.

(ii) $\lambda \cdot \mathbf{0} = \mathbf{0}$ and $0 \cdot \mathbf{v} = \mathbf{0}$.

(iii) $-(\lambda \mathbf{v}) = (-\lambda)\mathbf{v}$.

(T1.3) [$\mathbb{Z}_2^n$ as a vector space]

Consider $\mathbb{Z}_2 = \{0, 1\}$ with operations of addition and multiplication obtained from the ordinary integer operations through reduction modulo 2.
Provide the tables for addition and multiplication in $\mathbb{Z}_2$ and check out some basic arithmetical properties (cf. Exercise 1.1.5 in the lecture notes).
Is it clear how to turn $\mathbb{Z}_2^n := (\mathbb{Z}_2)^n$, the set of n-tuples over $\mathbb{Z}_2$, into a vector space?

(i) How can you visualize the space $\mathbb{Z}_2^3$?

(ii) What do the lines and planes (through the origin $\mathbf{0} = (0, 0)$) in $\mathbb{Z}_2^3$ look like?

(iii) How many lines (through the origin) are there in $\mathbb{Z}_2^3$ (in $\mathbb{Z}_2^n$)?

(T1.4) [$\mathbb{Z}_2^n$ and subsets]

One may consider the values 0 and 1 of $\mathbb{Z}_2$ as the logical values *true* and *false*, or as bits. The elements of $\mathbb{Z}_2^n$ then correspond to length n bit strings, as, e.g., $\mathbf{b} = 0100111$ for $n = 7$. A bit string $\mathbf{b} = b_1 \dots b_n$ of length n may also be regarded as an encoding of a subset $S_{\mathbf{b}}$ of the set $\{1, \dots, n\}$:

$$S_{\mathbf{b}} := \{i \colon b_i = 1\}.$$

(i) In $\mathbb{Z}_2$: show that addition and multiplication correspond to the logical operations `XOR` (exclusive or) and `AND`, respectively.

(ii) In $\mathbb{Z}_2^n$: how can you interpret vector addition and scalar multiplication in terms of length n bit strings, and in terms of subsets of $\{1, \dots, n\}$, respectively?