Fachbereich Mathematik
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TECHNISCHE UNIVERSITÄT DARMSTADT

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Linear Algebra I

Mock Exam

Please fill in the head of this problem sheet NOW and IN CLEAR CAPITAL LETTERS. Write your name on all sheets, and number them consecutively.

Family name: ____________________

First name: ____________________

Semester: ____________________

Course (e.g. MCS): ____________________

Student number: ____________________

exercise	1	2	3	4	5	6	total	mark
max. points	6	6	6	6	9	7	40	
points								

before handing in, please fold along this line

At the end of the exam, please fold this sheet once along the line above this paragraph in such a way that your name and the table of points remain visible, and put your answers between the two halves.
You are free to use books, lecture notes, notes of your own, problem sheets and their solutions. **Please do not use any electronic device.**
Remember that you are expected to give reasons for your results, and that most credit will be given for systematic solutions, not for the bare results.
Good luck!

Exercise 1 (6 Points)

(a) In the $\mathbb{R}$-vector space $\mathbb{R}^3$:

(i) Which of the following vectors extend the pair of vectors $(1,0,0)$, $(1,1,0)$ to a basis of $\mathbb{R}^3$?

$$(0,0,1), \quad (0,1,1), \quad (0,1,0).$$

(ii) Which of the vectors in the basis $B = ((1,1,0),(0,1,1),(1,0,1))$ can be exchanged for the vector $(1,2,1)$ so that we again obtain a basis of $\mathbb{R}^3$?

(b) Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and $\varphi, \psi : V \to V$ automorphisms. Which of the following are always automorphisms? Give proofs or counterexamples.

$$\psi^{-1}, \quad \varphi + \psi, \quad \varphi \circ \psi^{-1}.$$

Exercise 2 (6 Points)

Let V be an $\mathbb{F}$-vector space and $\varphi : V \to V$ a projection (i.e. $\varphi \circ \varphi = \varphi$). Show that

(a) $\mathrm{id}_V - \varphi$ is also a projection.

(b) $\ker(\varphi) = \mathrm{image}(\mathrm{id}_V - \varphi)$.

(c) $\mathrm{image}(\varphi) = \ker(\mathrm{id}_V - \varphi)$.

Exercise 3 (6 Points)

Consider the following system of linear equations in variables x, y and z with the parameter $k \in \mathbb{R}$. Determine for which k the system has no solution, a unique solution or more than one solution over $\mathbb{R}^3$.

$$\begin{array}{ccccccc} kx & + & y & + & z & = & 1 \\ x & + & ky & + & z & = & 1 \\ x & + & y & + & kz & = & 1 \end{array}$$

Exercise 4 (6 Points)

Consider

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

as matrices either over $\mathbb{R}$ or over $\mathbb{F}_2$.

(a) Determine the rank of A and B over $\mathbb{R}$ and $\mathbb{F}_2$.

(b) Provide inverses for A and B over $\mathbb{R}$ and $\mathbb{F}_2$ where possible.

Exercise 5 (9 Points)

Let φ be the linear map from $\mathbb{R}^3$ to $\mathbb{R}^4$ given by

$$\begin{aligned} \varphi(1,0,0) &= (1,-3,2,4), \\ \varphi(0,1,0) &= (5,-3,0,2), \\ \varphi(0,0,1) &= (-2,0,1,1). \end{aligned}$$

(a) Find the matrix representation A_φ of φ (w.r.t. the standard bases).

(b) Determine the kernel of φ.

(c) Determine the dimensions of $\ker(\varphi)$ and $\mathrm{image}(\varphi)$.

(d) Find bases for $\ker(\varphi)$ and $\mathrm{image}(\varphi)$.

(e) Determine all vectors $(x_1, x_2, x_3) \in \mathbb{R}^3$ with $\varphi(x_1, x_2, x_3) = (9, -9, 3, 9)$.

Exercise 6 (7 Points)

Consider a vector space V over a field $\mathbb{F}$ and subspaces U_1 and U_2 of V such that $V = U_1 \oplus U_2$. Further, let φ_i be an endomorphism of U_i, for $i = 1, 2$.

(a) Show that there is a unique endomorphism φ of V that extends φ_1 and φ_2 in the sense that $\varphi(\mathbf{u}) = \varphi_1(\mathbf{u})$ for all $\mathbf{u} \in U_1$ and $\varphi(\mathbf{u}) = \varphi_2(\mathbf{u})$ for all $\mathbf{u} \in U_2$.

(b) For $i = 1, 2$ let B_i be a labelled basis of U_i, $A_i = [\![\varphi_i]\!]_{B_i}^{B_i}$ the corresponding matrix representation of φ_i. Obtain a matrix representation A of φ (as in part (a)) w.r.t. a suitably chosen basis of V from A_1 and A_2.