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Solution Suggestion for the Mock Exam
Linear Algebra I for MCS
Winter Term 2006/2007

(Exercise.1)

(a) (i) Clearly the span of the two given vectors is $\text{span}(\mathbf{e}_1, \mathbf{e}_2)$; therefore any vector not contained in $\text{span}(\mathbf{e}_1, \mathbf{e}_2)$ will do, i.e., all apart from the last.

(ii) Now, $(1, 2, 1) = (1, 1, 0) + (0, 1, 1)$, but it is impossible to express $(1, 2, 1)$ as a linear combination of either the second and third, or the first and third (as a quick calculation shows). So only replacing the third by $(1, 2, 1)$ would no longer give a basis.

(b) The map ψ^{-1} is again a bijection and linear, hence an automorphism (cf. tutorial exercise T9.1 (i).

The map $\varphi + \psi$ is again linear, but not necessarily an automorphism. Consider, for example, $\varphi = \text{id} : \mathbb{R} \to \mathbb{R}$ and $\psi = -\text{id} : \mathbb{R} \to \mathbb{R}$.

$\varphi \circ \psi^{-1}$ is, by the above, the composite of two automorphisms, and automorphisms are closed under composition, since both bijections and linear maps are.

(Exercise.2)

(a) By direct computation:

$$(\text{id}_V - \varphi) \circ (\text{id}_V - \varphi) = \text{id}_V \circ \text{id}_V - \text{id}_V \circ \varphi - \varphi \circ \text{id}_V + \varphi \circ \varphi = \text{id}_V - \varphi - \varphi + \varphi = \text{id}_V - \varphi.$$

(b) We show both inclusions.
$\ker(\varphi) \subseteq \text{image}(\text{id}_V - \varphi)$: If $\mathbf{v} \in \ker(\varphi)$, then $\varphi(\mathbf{v}) = \mathbf{0}$, so $(\text{id}_V - \varphi)(\mathbf{v}) = \mathbf{v}$. Therefore $\mathbf{v} \in \text{image}(\text{id}_V - \varphi)$.
$\text{image}(\text{id}_V - \varphi) \subseteq \ker(\varphi)$: If $\mathbf{v} \in \text{image}(\text{id}_V - \varphi)$, then there is a $\mathbf{w} \in V$ such that $\mathbf{w} - \varphi(\mathbf{w}) = \mathbf{v}$. Applying φ to both sides and using that $\varphi \circ \varphi = \varphi$, we see that $\varphi(\mathbf{v}) = \varphi(\mathbf{w}) - \varphi(\varphi(\mathbf{w}))\varphi(\mathbf{w}) - \varphi(\mathbf{w}) = \mathbf{0}$. So $\mathbf{v} \in \ker(\varphi)$.

(c) follows from the previous two by applying (b) to the map $\mathrm{id}_V - \varphi$, which is a projection by (a).

(Exercise.3)

To solve

$$\begin{array}{ccccccc} kx & + & y & + & z & = & 1 \\ x & + & ky & + & z & = & 1 \\ x & + & y & + & kz & = & 1 \end{array},$$

we first change the order of the rows and then further apply Gauß-Jordan elimination to bring the matrix into echelon form.

$$\left.\begin{array}{ccc|c} 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \\ k & 1 & 1 & 1 \end{array}\right. \rightsquigarrow \left.\begin{array}{ccc|c} 1 & k & 1 & 1 \\ 1 & 1-k & k-1 & 0 \\ 0 & (1-k)(1+k) & 1-k & 1-k \end{array}\right. \rightsquigarrow \left.\begin{array}{ccc|c} 1 & k & 1 & 1 \\ 0 & 1-k & k-1 & 0 \\ 0 & 0 & (1-k)(k+2) & 1-k \end{array}\right. .$$

This means that there are no solutions when $k = -2$, and infinitely many solutions when $k = 1$. When $k \neq -2, 1$, there is precisely one solution.

(Exercise.4)

(i) Let us first consider the matrix A. Subtracting the first row from the second and the third we obtain

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ for } \mathbb{R} \text{ and } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ for } \mathbb{F}_2,$$

so we have $\operatorname{rank} A = 3$ for $\mathbb{R}$ and $\operatorname{rank} A = 2$ for $\mathbb{F}_2$.

By adding the second row of B to the third and then exchanging the second and the first, we get

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

so B has rank 3 for both fields.

(ii) With a few steps of Gauß-Jordan one computes that

$$A^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 2 & 1 \\ 2 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix},$$

which is true for $\mathbb{R}$. For the field $\mathbb{F}_2$, we know that A is not invertible and

$$B^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

(Exercise.5)

(i) We simply get

$$A = [\![\varphi]\!]^{S_3}_{S_4} = \begin{pmatrix} 1 & 5 & -2 \\ -3 & -3 & 0 \\ 2 & 0 & 1 \\ 4 & 2 & 1 \end{pmatrix},$$

where S_3 and S_4 are the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.

(ii) To solve $Ax = \mathbf{0}$ we directly read from the second row that $x_1 = -x_2$ and from the third row that $x_3 = -2x_1$. So we have $\{\lambda(-1, 1, 2) : \lambda \in \mathbb{R}\} \supseteq \ker(\varphi)$ and one directly verifies that this is indeed the kernel.

(iii) From (ii) we conclude $\dim(\ker(\varphi)) = 1$ which by the dimension formula implies that $\dim(\mathrm{image}(\varphi)) = 2$.

(iv) From (ii) we get that $\{(-1, 1, 2)\}$ is a basis of $\ker(\varphi)$. Since we know that $\dim(\mathrm{image}(\varphi)) = 2$ we can just take two linear independent rows of A. Obviously, each choice works.

(v) We do Gauß-Jordan elimination:

$$\begin{array}{ccc|c} 1 & 5 & -2 & 9 \\ -3 & -3 & 0 & -9 \\ 2 & 0 & 1 & 3 \\ 4 & 2 & 1 & 9 \end{array} \rightsquigarrow \begin{array}{ccc|c} 1 & 5 & -2 & 9 \\ 0 & 12 & -6 & 18 \\ 0 & -10 & 5 & -15 \\ 0 & -18 & 9 & -27 \end{array} \rightsquigarrow \begin{array}{ccc|c} 1 & 5 & -2 & 9 \\ 0 & 12 & -6 & 18 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

Setting $x_2 = 0$ we get that one solution is $(3, 0, -3)$ and so all solutions are given by $(3, 0, -3) + \ker(\varphi) = \{(3 - \lambda, \lambda, 2\lambda - 3) : \lambda \in \mathbb{R}\}$.

(Exercise.6)

(i) Take labelled bases B_1 of U_1 and B_2 of U_2. Then $B = (B_1, B_2)$ is a labelled basis of $V = U_1 \oplus U_2$. By Proposition 3.1.8 φ is uniquely determined by the images of all $\mathbf{b} \in B_1$ and $\mathbf{b}_2 \in B_2$, respectively. So we just take the images under φ_1 or φ_2, respectively, to specify φ. Clearly φ is as desired, and unique determined by these necessary stipulations (Proposition 3.1.8).

(ii) Of course we take B as our basis and get a representation in "block diagonal" form

$$A = \left(\begin{array}{c|c} A_1 & \mathbf{0} \\ \hline \mathbf{0} & A_2 \end{array}\right).$$