**Fachbereich Mathematik**
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# 14th Exercise Sheet Linear Algebra I for MCS
## Winter Term 2006/2007

**(E14.1) [Scalar product]**

The *standard scalar product* [Standardskalarprodukt] on $\mathbb{F}^n$ is defined to be the map

$$\langle \cdot, \cdot \rangle : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}, \quad (\mathbf{x}, \mathbf{y}) \mapsto \langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^{n} x_i y_i.$$

Show that $\langle \cdot, \cdot \rangle$ has the following properties:

(i) it is bilinear,

(ii) it is symmetric (i.e. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$),

(iii) $\langle \mathbf{x}, \mathbf{e}_i \rangle = x_i$ for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}^n$ and $1 \leqslant i \leqslant n$.

Two vectors $\mathbf{x}$ and $\mathbf{y}$ are called *orthogonal* [orthogonal], $\mathbf{x} \perp \mathbf{y}$, if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

**(E14.2) [Cross product]**

Let $\mathbb{F}^3$ be equipped with the standard scalar product (as above). Consider a function

$$\times : \mathbb{F}^3 \times \mathbb{F}^3 \to \mathbb{F}^3, \quad (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \times \mathbf{y},$$

with the following properties:

- bi-linearity (linearity in each argument, as an $\mathbb{F}^3$-valued function),
- antisymmetry (i.e., for all $\mathbf{x} \in \mathbb{F}^3$, $\mathbf{x} \times \mathbf{x} = \mathbf{0}$),
- $\mathbf{x} \times \mathbf{y}$ is orthogonal to $\mathbf{x}$ and $\mathbf{y}$,
- normalization: $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$.

The aim of this exercise is to show that a function with these properties is uniquely determined, and to give an explicit description of this function.

(i) Prove that the function $g : \mathbb{F}^3 \times \mathbb{F}^3 \times \mathbb{F}^3 \to \mathbb{F}, \quad (\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto \langle \mathbf{x} \times \mathbf{y}, \mathbf{z} \rangle$ is 3-linear, antisymmetric and normalized (in the sense that $g(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 1$).

(ii) Conclude that $g$ and hence $\times$ is uniquely determined.

(iii) Use E14.1(iii) to give an explicit formula for $\mathbf{x} \times \mathbf{y}$ in terms of $x_i, y_i$, and show that the function thus obtained satisfies all requirements.

(iv) Show that $\times$ is not associative.

**(E14.3) [Symmetries of patterns]**

Consider a subgroup $H$ of the group of affine transformations of $\mathbb{R}^2$ (cf. section 3.4 of the notes) and let

$$\begin{aligned} H_0 &:= \{\varphi \in \mathrm{GL}_2(\mathbb{R})\colon [\varphi, \mathbf{u}] \in H \text{ for some } \mathbf{u} \in \mathbb{R}^2\} \subseteq \mathrm{GL}_2(\mathbb{R}); \\ L_H &:= \{\mathbf{u} \in \mathbb{R}^2\colon \tau_{\mathbf{u}} \in H\} \subseteq \mathbb{R}^2. \end{aligned}$$

(i) Show that $H_0$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$. Give an example that not necessarily $H_0 \subseteq H$.

(ii) Show that $L_H$ is closed under vector addition (a subgroup of $(\mathbb{R}^2, +, \mathbf{0})$) and that $L_H$ is mapped onto itself by $H_0$.

(iii) $(*)$ Assuming that $L_H \neq \{\mathbf{0}\}$ does no contain arbitrarily small non-zero vectors, show that $L_H$ is the set of integer linear combinations of one or two vectors in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ (a one- or two-dimensional lattice).

(iv) Try to describe the symmetry group $H$ of the pattern below as a subgroup of the group of affine transformations. Determine $L_H$ and $H_0$ and try to describe how these fit together in $H$. Can you find a small finite set of transformations in $H$ that generate $H$ in the sense that all elements of $H$ are compositions of the given ones?