

- (ii) Let now $x_i \in \mathbb{F}$ for $1 \leq i \leq n$. Determine for which $(x_1, \dots, x_n) \in \mathbb{F}^n$ the following matrix A_n is regular.

$$A_n = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix}.$$

Hint: Use induction on n to show that $|A_n| = \prod_{1 \leq i < j \leq n} (x_i - x_j)$.

- (iii) (*) Let $x_i, y_i \in \mathbb{R}$ for $1 \leq i \leq n$ such that the x_i are pairwise distinct. Prove that there exists a unique polynomial $f(x)$ of degree $n - 1$ such that $f(x_i) = y_i$.

(E13.4) [Transformation into a diagonal matrix]

Consider the linear map φ from \mathbb{R}^3 to \mathbb{R}^3 with the matrix representation

$$A = [\varphi]_S^S = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

w.r.t the standard basis S .

- (i) Show that $\mathbf{x} = (1, 1, 0)$ and $\mathbf{y} = (1, 0, -1)$ are solutions for $A\mathbf{v} = -2\mathbf{v}$.¹
- (ii) Show that $\mathbf{z} = (1, 1, 2)$ solves the system of linear equations $A\mathbf{v} = 4\mathbf{v}$.
- (iii) Show that \mathbf{x}, \mathbf{y} and \mathbf{z} are linearly independent.
- (iv) Determine the coefficient of $\mathbf{e}_1, \mathbf{e}_2$ and \mathbf{e}_3 with respect to the basis $B = (\mathbf{x}, \mathbf{y}, \mathbf{z})$.
- (v) Find $[\varphi]_B^B$ and determine $\det([\varphi]_B^B)$ both via the old and via the new matrix representation.

(E13.5) [Calculating the inverse]

Consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

with real entries.

- (i) Solve the following linear system of equations : $A\mathbf{x} = \mathbf{y} = (y_1, y_2, y_3)$ over \mathbb{R} .
- (ii) Determine \mathbf{x} for $\mathbf{y} = \mathbf{e}_1, \mathbf{y} = \mathbf{e}_2$ and $\mathbf{y} = \mathbf{e}_3$
- (iii) Calculate the inverse of A using part (ii).

¹In this case one calls \mathbf{x} and \mathbf{y} *eigenvectors* of A (or φ) for the *eigenvalue* -2 .