Fachbereich Mathematik
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13th Exercise Sheet Linear Algebra I for MCS
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(E13.1) [Determinant of a large matrix]

Calculate the determinant of the $n \times n$-matrix

$$A_n = \begin{pmatrix} 1 & 1 & & & & 0 \\ -2 & -1 & 1 & & & \\ & -2 & -1 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -2 & -1 & 1 \\ 0 & & & & -2 & -1 \end{pmatrix}.$$

Proceed inductively via row or column expansion.

(E13.2) [Transpose matrix]

(i) Let A be an arbitrary matrix. Under what conditions is the product AA^T defined?

(ii) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$. Find AA^T and A^TA.

(iii) Prove that $(AB)^T = B^TA^T$.

(iv) A matrix A is called symmetric if $A = A^T$. Prove that AA^T is symmetric for arbitrary matrices A.

(E13.3) [Vandermonde matrix]

(i) Let $\mathbb{F}$ be a field and $x_1, x_2, x_3 \in \mathbb{F}$. Calculate the determinant of the matrices

$$A_2 = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix},$$

and prove that A_3 is regular if, and only if, the x_i are pairwise distinct.

(ii) Let now $x_i \in \mathbb{F}$ for $1 \leqslant i \leqslant n$. Determine for which $(x_1, \dots, x_n) \in \mathbb{F}^n$ the following matrix A_n is regular.

$$A_n = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix}.$$

Hint: Use induction on n to show that $|A_n| = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)$.

(iii) (*) Let $x_i, y_i \in \mathbb{R}$ for $1 \leqslant i \leqslant n$ such that the x_i are pairwise distinct. Prove that there exists a unique polynomial $f(x)$ of degree $n-1$ such that $f(x_i) = y_i$.

(E13.4) [Transformation into a diagonal matrix]

Consider the linear map φ from $\mathbb{R}^3$ to $\mathbb{R}^3$ with the matrix representation

$$A = [\![\varphi]\!]_S^S = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

w.r.t the standard basis S.

(i) Show that $\mathbf{x} = (1, 1, 0)$ and $\mathbf{y} = (1, 0, -1)$ are solutions for $A\mathbf{v} = -2\mathbf{v}$.[1]

(ii) Show that $\mathbf{z} = (1, 1, 2)$ solves the system of linear equations $A\mathbf{v} = 4\mathbf{v}$.

(iii) Show that $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ are linearly independent.

(iv) Determine the coefficient of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ with respect to the basis $B = (\mathbf{x}, \mathbf{y}, \mathbf{z})$.

(v) Find $[\![\varphi]\!]_B^B$ and determine $\det([\![\varphi]\!]_B^B)$ both via the old and via the new matrix representation.

(E13.5) [Calculating the inverse]

Consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

with real entries.

(i) Solve the following linear system of equations : $A\mathbf{x} = \mathbf{y} = (y_1, y_2, y_3)$ over $\mathbb{R}$.

(ii) Determine $\mathbf{x}$ for $\mathbf{y} = \mathbf{e}_1, \mathbf{y} = \mathbf{e}_2$ and $\mathbf{y} = \mathbf{e}_3$

(iii) Calculate the inverse of A using part (ii).

[1] In this case one calls $\mathbf{x}$ and $\mathbf{y}$ *eigenvectors* of A (or φ) for the *eigenvalue* -2.