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12th Exercise Sheet Linear Algebra I for MCS
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(E12.1) [Geometry and determinants]

Based on the geometric intuition of det as an oriented volume in $\mathbb{R}^3$, what should the determinants of the (matrix representations of the) following be?

(i) The rotation $\delta : \mathbb{R}^3 \to \mathbb{R}^3$ through some angle α about some axis.

(ii) The reflection $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ in some plane.

Verify these intuitions in terms of matrix representations.

(E12.2) [Determinants]

(i) Let A and B be two real-valued matrices

$$A = \begin{pmatrix} -2 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & 2 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 3 \\ 2 & -1 & 4 \end{pmatrix}$$

Calculate $\det(A)$, $\det(B)$, $\det(AB)$, $\det(BA)$ and $\det(A+B)$. Which of these matrices are invertible?

(ii) Let A and B be the two matrices

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 2 & 1 & 4 \\ 4 & 2 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{pmatrix}$$

over the field $\mathbb{F}_5$. Calculate the values of $\det(A)$, $\det(B)$, $\det(AB)$, $\det(BA)$ and $\det(A + B)$. Which of these matrices are invertible in $\mathrm{GL}_3(\mathbb{F}_5)$? Compare with the first part of the exercise!

(E12.3) [Linear maps and determinants]

(i) Find $\det(\varphi)$ for $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ with $\varphi(x, y, z) = (2x - z, x + 2y - 4z, 3x - 3y + z)$.

(ii) Consider the vector space $V = \mathbb{F}^{(2,2)}$ of 2×2 matrices over $\mathbb{F}$ and the linear map

$$\nu : \mathbb{F}^{(2,2)} \to \mathbb{F}^{(2,2)},$$
$$A \mapsto M \cdot A \text{ (matrix product)}$$

for a fixed matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{F}^{(2,2)}$.

(a) Prove that ν is linear.

(b) Give a basis B for V.

(c) Give $[\![\nu]\!]_B^B$.

(d) Determine the determinant of ν.