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11th Exercise Sheet Linear Algebra I for MCS
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(E11.1) [Zero divisors]

(i) Let $A \in \mathbb{F}^{(n,n)}$ be some matrix. Show that A is not invertible if, and only if, there exists some matrix $B \in \mathbb{F}^{(n,n)}$, $B \neq \mathbf{0}$ with $AB = \mathbf{0}$.

(*Hint*: use the fact that a matrix represents a linear map and choose a suitable basis.)

(ii) (*) Let now R be a ring and $a \in R$. Which directions of the statement "a is not invertible if, and only if, there exists some $b \in R$ with $ab = 0$." still hold? Give proof(s) or counterexample(s).

(E11.2) [Calculating with block matrices]

(i) Let $A_{11} \in \mathbb{F}^{(n,p)}$, $A_{12} \in \mathbb{F}^{(n,q)}$, $B_{11} \in \mathbb{F}^{(p,m)}$ and $B_{21} \in \mathbb{F}^{(q,m)}$. Show that the product of the block matrices can be calculated as

$$\begin{array}{|c|c|}\hline A_{11} & A_{12} \\ \hline\end{array} \cdot \begin{array}{|c|}\hline B_{11} \\ \hline B_{21} \\ \hline\end{array} = \begin{array}{|c|}\hline A_{11}B_{11} + A_{12}B_{21} \\ \hline\end{array}\,.$$

What is the size of the resulting matrix?

(ii) Let A and B be two $n \times n$ block matrices of the form

$$A = \begin{array}{|c|c|}\hline A_{11} & A_{12} \\ \hline 0 & A_{22} \\ \hline\end{array}\,, \qquad B = \begin{array}{|c|c|}\hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline\end{array}\,,$$

where $A_{11}, B_{11} \in \mathbb{F}^{(p,p)}$, $A_{22}, B_{22} \in \mathbb{F}^{(n-p,n-p)}$, etc.

Calculate the product of A and B in blocks:

$$\begin{array}{|c|c|}\hline A_{11} & A_{12} \\ \hline 0 & A_{22} \\ \hline\end{array} \cdot \begin{array}{|c|c|}\hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline\end{array} = \begin{array}{|c|c|}\hline ? & ? \\ \hline ? & ? \\ \hline\end{array}\,.$$

(E11.3) [Projections with respect to a suitable basis]

Let $V = U \oplus W$ be a finite dimensional vector space, and $\pi : V \to V$ a projection (i.e. $\pi \circ \pi = \pi$, cf. T9.2) such that $U = \ker(\pi)$ and $W = \mathrm{image}(\pi)$.

(i) Show that there exists a basis of V such that the matrix of π with respect to that basis is a diagonal matrix with only ones and zeroes on the diagonal.[1]

Hint: start from bases of U and W, respectively.

(ii) Let $\pi : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the matrix

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 4 & 5 & -4 \\ 4 & 4 & -3 \end{pmatrix}$$

with respect to the standard basis. Verify that π is a projection. Determine a matrix S and a diagonal matrix D such that $D = S^{-1}AS$.

(E11.4) [Basis transformations]

(i) The linear map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is given by the matrix

$$[\![\varphi]\!]_E^E = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

w.r.t. the standard basis E.

Compute the matrix $[\![\varphi]\!]_B^B$ of φ w.r.t. the basis $B = ((1,-1,-1),(1,0,1),(1,1,1))$.

(ii) We consider the $\mathbb{R}$-vector spaces $\mathbb{R}^2$ und $\mathbb{R}^3$ with bases $B = ((0,1),(1,0))$ and $C = ((0,1,1),(1,0,1),(1,1,0))$. A linear map $\psi \in \mathit{Hom}(\mathbb{R}^3, \mathbb{R}^2)$ is given by

$$[\![\psi]\!]_B^C := \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

Derive the matrix of ψ w.r.t. the standard bases S_2, S_3.

[1] Compare this **carefully** to the representation of an arbitrary endomorphism in E9.3(iii). What are the similarities/differences?