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December 21, 2006

10th Exercise Sheet Linear Algebra I for MCS
Winter Term 2006/2007

(E10.1) [Dual basis]

(Compare section 3.2.2 in the notes and Observation 3.2.6 in particular.)
Let $V = \mathbb{F}^2$ and $V^* = \mathrm{Hom}(V, \mathbb{F})$ its dual space. Further, let V be equipped with the labelled basis $B = ((1,1),(1,2))$. Determine the dual basis B^* of V^* corresponding to B.

(E10.2) [Matrix powers]

Determine all powers A^n, for $n \in \mathbb{N}$, for these matrices:

$$\begin{pmatrix} 0 & 3 \\ 1/3 & 0 \end{pmatrix}, \begin{pmatrix} -6 & 5 \\ -7 & 6 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & b & 0 & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(E10.3) [Trace]

The *trace* [Spur] of a matrix is defined as the map

$$\mathrm{trace} : \mathbb{F}^{(n,n)} \to \mathbb{F}, \quad A = (a_{ij}) \mapsto \sum_{i=1}^{n} a_{ii}.$$

(i) Show that trace is linear.

(ii) Determine the dimension of image(trace) and ker(trace) and give a basis for ker(trace) for $n = 2$ (the space of 2×2 matrices).[1]

(iii) Show that $\mathrm{trace}(AB) = \mathrm{trace}(BA)$ for all $A, B \in \mathbb{F}^{(2,2)}$.[2]

[1] You may want to do this and the rest of the exercise for arbitrary n instead of $n = 2$, which is not much harder.

[2] It follows that $\mathrm{trace}(S^{-1}AS) = \mathrm{trace}(A)$ for all regular matrices S. As we shall see later, this implies that trace is invariant under basis transformation. So one can define the trace of an endomorphism φ just as $\mathrm{trace}(\varphi) = \mathrm{trace}[\![\varphi]\!]_B^B$ where B is any basis!

(iv) (∗) Let $\varphi \in \mathrm{Hom}(\mathbb{F}^{(2,2)}, \mathbb{F})$ such that $\varphi(AB) = \varphi(BA)$ for all $A, B \in \mathbb{F}^{(2,2)}$.
Show that there exists an element $c \in \mathbb{F}$ such that $\varphi = c\,\mathrm{trace}$.[3]

Hint: use the fact that $AB - BA \in \ker(\varphi)$ for all $A, B \in \mathbb{F}^{(2,2)}$, and first show that $\ker(\varphi) = \ker(\mathrm{trace})$.

(E10.4) [Nilpotent endomorphisms]

Let V be an $\mathbb{F}$-vector space and $\varphi : V \to V$ an endomorphism. Prove:

(i) $V \supseteq \mathrm{image}(\varphi) \supseteq \mathrm{image}(\varphi^2) \supseteq \mathrm{image}(\varphi^3) \supseteq \cdots$

(ii) $\{\mathbf{0}\} \subseteq \ker(\varphi) \subseteq \ker(\varphi^2) \subseteq \ker(\varphi^3) \subseteq \cdots$

(iii) An endomorphism $\varphi : V \to V$ is called *nilpotent* if $\varphi^m = \mathbf{0}$ for some $m \in \mathbb{N}$.
Let $\varphi \neq \mathbf{0}$ be a nilpotent endomorphism of V and $m \in \mathbb{N}$ such that $\varphi^{m-1} \neq \mathbf{0}$ and $\varphi^m = \mathbf{0}$. Show that $\{\mathbf{0}\} \subsetneq \ker(\varphi) \subsetneq \ker(\varphi^2) \subsetneq \cdots \subsetneq \ker(\varphi^m) = V$.

[3] So the property in (iii) determines trace uniquely up to a scalar.

Christmas Exercises:

(E10.5) [Systems of linear equations and linear maps]

For a matrix $A = (a_{ij}) \in \mathbb{F}^{(m,n)}$ with column vectors $\mathbf{a}_j$, $1 \leqslant j \leqslant n$ and another column vector $\mathbf{b} \in \mathbb{F}^m$,

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix},$$

consider the following system of m linear equations over $\mathbb{F}^n$

$$E = E[A, \mathbf{b}]\colon \left\{ \begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & +\cdots+ & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & +\cdots+ & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & +\cdots+ & a_{3n}x_n & = & b_3 \\ & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & +\cdots+ & a_{mn}x_n & = & b_m, \end{array} \right.$$

and the associated linear map $\varphi_A\colon \mathbb{F}^n \to \mathbb{F}^m$ given as $\varphi_A(x_1, \ldots, x_n) = \sum_{i=1}^n x_j \mathbf{a}_j$.

Explore and explain as many correspondences and connections as you can between the following features/conditions/parameters (also in relation to n and m):

- $\ker(\varphi_A)$ and its dimension.
- $\text{image}(\varphi_A)$ and its dimension.
- $\text{span}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ and its dimension.
- $\text{span}(\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b})$ and its dimension.
- $S(E)$ and its dimension.
- $S(E^*)$ and its dimension.
- injectivity of φ_A.
- surjectivity of φ_A.
- $\mathbf{b} \in \text{image}(\varphi_A)$.
- solvability of $E[A, \mathbf{b}]$.
- unique solvability of $E[A, \mathbf{b}]$.
- solvability of $E[A, \mathbf{b}']$ for all $\mathbf{b}' \in \mathbb{F}^m$.
- unique solvability of $E[A, \mathbf{b}']$ for all $\mathbf{b}' \in \mathbb{F}^m$.

It may help to test these in concrete examples. You may take some of those below.

(i) $\mathbb{F} = \mathbb{R}$ and

$$A = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix},$$

(ii) $\mathbb{F} = \mathbb{F}^2$ and

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

(iii) $\mathbb{F} = \mathbb{C}$ and

$$A = \begin{pmatrix} 1 & i & -1 & -i \\ 1 & 2 & 4 & 8 \\ 1 & \sqrt{2} & 2 & 2\sqrt{2} \\ 1 & 1+i & 2i & 2i-2 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 25i + 3 \\ \sqrt{2} - i \\ \pi \\ \sqrt{11} - 1.33i \end{pmatrix}.$$

(E10.6) [Hamming code, error correction, lie detection][4]

Consider the system of linear equations over $\mathbb{F}_2$ associated to the following matrix $A \in \mathbb{F}_2^{(3,7)}$ with induced linear map $\varphi\colon \mathbb{F}_2^7 \to \mathbb{F}_2^3$:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \varphi(x_1, \ldots, x_7) := A \begin{pmatrix} x_1 \\ \vdots \\ x_7 \end{pmatrix}.$$

(i) Solve the system of homogeneous equations induced by A in order to determine $\ker(\varphi)$. What are the dimensions and cardinalities of $\ker(\varphi)$ and $\text{image}(\varphi)$?

(ii) Consider $C = \ker(\varphi) \subseteq \mathbb{F}_2^7$ as the set of admissible codes. Explain how the ability to identify and correct errors involving up to one bit in the transmission of $\mathbf{c} \in C$ precisely corresponds to the requirement that φ is injective in restriction to the subset (*not* a subspace!)

$$\mathbf{0}^{\sim} := \{\mathbf{0}\} \cup \{\mathbf{e}_1, \ldots, \mathbf{e}_7\},$$

the set of all vectors at Hamming distance up to 1 from $\mathbf{0}$.
Check that φ satisfies this condition by checking that $\varphi(\mathbf{e}_i) = \langle i \rangle_2$.[5]

[4] Compare the second tutorial sheet, T2.2 and T2.3. The system of linear equations given by the matrix A below differs from the one in T2.2 just by a rearrangement of the rows which is more convenient here.

[5] $\langle n \rangle_2$ stands for the binary representation of the number n.

(iii) Describe a simple (algorithmic) procedure based on φ which on inputs from $\tilde{C} := C + \mathbf{0}^{\sim} = C + \{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_7\}$ (codes disturbed in up to one bit) reconstructs the nearest member from C (the original, unperturbed code).

(iv) (∗) As an extra, suggest how to extend the coding scheme explored above

(a) to allow for larger C (more admissible codes). E.g., can you determine the least n for which there is a subspace $C' \subseteq \mathbb{F}_2^n$ with at least 32 elements and allowing for the correction of one error?

(b) to allow for more than one error. E.g., suggest a similar set-up that can deal with up to two errors. In particular, can you determine the least n for which there is such a subspace $C' \subseteq \mathbb{F}_2^n$ with at least 16 elements?
Hint: instead of $\mathbf{0}^{\sim}$ consider

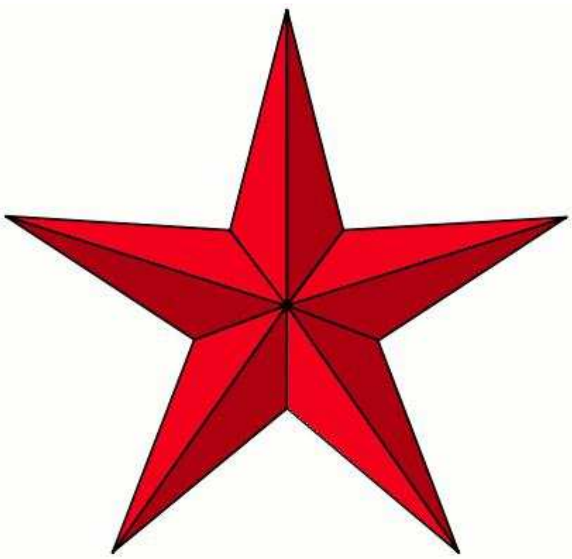
$$\mathbf{0}^{\approx} = \{\mathbf{0}\} \cup \{\mathbf{e}_i : 1 \leqslant i \leqslant n\} \cup \{\mathbf{e}_i + \mathbf{e}_j : 1 \leqslant i < j \leqslant n\}$$

and note that φ is injective on this subset iff no sum of up to four vectors $\varphi(\mathbf{e}_i)$ for distinct i evaluates to $\mathbf{0}$.

An alternative interpretation. Consider any injection $\psi\colon \{0, \ldots, N-1\} \to C \subseteq \mathbb{F}_2^n$. Let $\psi(m) = (\psi_1(m), \ldots, \psi_n(m))$ and interpret each component map ψ_i as a yes/no question about numbers $m \in \{0, \ldots, N-1\}$ according to

Does m belong to the subset $\{q < N\colon \psi_i(q) = 0\}$?

For suitable C as above, one can extract m from the sequence of answers to these n questions about m, even if the respondent is allowed to lie once, and the lie can therefore be identified. You may want to suggest, for $n = 7$ and $N = 16$, a concrete set of 7 questions based on the above C, some linear map $\xi\colon \mathbb{F}_2^4 \to \mathbb{F}_2^7$, and $\psi(m) := \xi(\langle m \rangle_2)$.



(*) Consider the star on the left as a subset S of $\mathbb{R}^2$ (with the center of the star as the origin $\mathbf{0}$). For which linear maps $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is the star mapped onto itself ($\varphi(S) = S$)? If you want, you can also consider stars with an arbitrary number of spikes, and explore the structure of its symmetry group.

The Linear Algebra I team wishes you a Merry Christmas and a Happy New Year 2007!