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9th Exercise Sheet Linear Algebra I for MCS
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(E9.1) [Endomorphisms]

(i) Let V be a vector space and $\varphi : V \to V$ a vector space endomorphism. Prove that $\mathrm{image}(\varphi) \subseteq \ker(\varphi)$ if and only if $\varphi \circ \varphi = \mathbf{0}$ (the null map, constant with value $\mathbf{0}$).

(ii) Find a linear map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\ker(\varphi) = \mathrm{image}(\varphi)$.

(E9.2) [Matrices for endomorphisms]

(i) Consider the endomorphism of $\mathbb{R}^2$ corresponding to the reflection in the line through $\mathbf{0}$ that is spanned by the vector $\mathbf{x} = (2, 1)$. Obtain its matrix representation with respect to the standard basis.

Hint: Draw a sketch and construct the images of the vectors $(5, 0)$ and $(0, 5)$.

(ii) Can you think of a basis of $\mathbb{R}^2$ with respect to which the matrix looks much simpler? Can you generalize this to arbitrary reflections?

(iii) Now consider the rotation in $\mathbb{R}^3$ through 120 degrees about the axis through $\mathbf{0}$ spanned by the vector $\mathbf{x} = (1, 1, 1)$. Draw a sketch of the situation and obtain the matrix representation with respect to a suitable basis.

(E9.3) [Matrices and linear maps]

Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map such that $\varphi(x, y, z) = (x + y + z, x + 2z, x + 2y)$.

(i) Determine $\dim(\ker(\varphi))$.

(ii) Find bases B_1, B_2 of $\mathbb{R}^3$ such that $\quad [\![\varphi]\!]_{B_2}^{B_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(iii) Let $\varphi : V \to V$ be an endomorphism of an arbitrary n-dimensional vector space. Prove that there exists bases B_1, B_2 of V, such that

$$[\![\varphi]\!]^{B_1}_{B_2} = \left(\begin{array}{c|c} E_r & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right),$$

where $r = \dim(\text{image}(\varphi))$ and E_r is the $r \times r$-unit matrix.

(E9.4) [An alternative proof of the dimension formula]

(i) Let $\varphi : V \to W$ be a linear map. Prove that $V/\ker(\varphi)$ is isomorphic to $\text{image}(\varphi)$.

(ii) Use this result to give an alternative proof of the dimension formula.

(E9.5) [The vector space $Hom(V, W)$**]**

(This is Exercise 3.2.1 in the notes.) Let $\mathbb{F}$ be a field, V and W $\mathbb{F}$-vector spaces with labelled bases $(\mathbf{a}_1, \dots \mathbf{a}_n)$, $(\mathbf{b}_1, \dots, \mathbf{b}_m)$, respectively. Consider the maps φ_{ij} for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ defined by

$$\varphi_{ij} \colon \begin{array}{rcl} V & \longrightarrow & W \\ \sum_{\ell=1}^{n} \lambda_\ell \mathbf{a}_\ell & \longmapsto & \lambda_j \mathbf{b}_i. \end{array}$$

(i) Prove that the maps φ_{ij} are linear and pairwise distinct.

(ii) Prove that the maps φ_{ij} are linearly independent and span $Hom(V, W)$. Use this to determine the dimension of $Hom(V, W)$.

Hint: For linear independence observe that a linear map (here: a linear combination of the φ_{ij}) is the null map iff it has value $\mathbf{0}$ on all arguments (equivalently: on all basis vectors). Similarly for a representation of an arbitrary $\varphi \in \text{Hom}(V, W)$ as a linear combination of the φ_{ij}, use that two linear maps are equal if they agree on all basis vectors. The rest is matching of coefficients.