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8th Exercise Sheet Linear Algebra I for MCS
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(E8.1) [Direct sums]

(i) Let $\mathbb{F}$ a field, V a finite-dimensional vector space and $U \subseteq V$ a subspace. Show that there exists a subspace W such that V is the direct sum of U and W (a *linear complement* for U, $V = U \oplus W$).

(ii) Let now V be the direct sum of U and W, $V = U \oplus W$. Show that if $A = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subseteq U$ and $B = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\} \subseteq W$ are linearly independent, then so is $A \cup B$,

(E8.2) [Linear maps]

(i) Which of the following functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$ are linear? Justify your answers!

$$\varphi_1(x_1, x_2) = x_1 + x_2 \qquad \varphi_2(x_1, x_2) = x_1 \cdot x_2$$
$$\varphi_3(x_1, x_2) = -x_2 \qquad \varphi_4(x_1, x_2) = -x_1 + 2$$

(ii) Which of the following functions $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ are linear? Which are invertible? Justify your answers!

$$\varphi_1(x_1, x_2) = (x_1 - 2, 3x_2) \qquad \varphi_2(x_1, x_2) = (2x_1 + x_2, x_2 - x_1)$$
$$\varphi_3(x_1, x_2) = (2x_1 - 3x_2, 0) \qquad \varphi_4(x_1, x_2) = (x_1^2, 2x_2)$$

(E8.3) [Polynomials]

Consider the map $\varphi : \mathrm{Pol}(\mathbb{R}) \to \mathrm{Pol}(\mathbb{R}), p \mapsto p'$, where p' is the derivative of the polynomial function p. [For $p = \sum_{i=0}^{n} a_i x^i$, $p' = \sum_{i=1}^{n} i a_i x^{i-1}$.]

(i) Is φ linear?

(ii) What is the kernel of φ?

(iii) What is the image of φ?

(iv) Is φ injective? Is it surjective?

(E8.4) [Affine subspaces]

Let $\mathbb{F}$ be a field, V,W $\mathbb{F}$-vector spaces and $\varphi : V \to W$ a linear map. Prove that if B is an affine subspace of W, its preimage $\varphi^{-1}(B) = \{x \in V : \varphi(x) \in B\}$ is an affine subspace of V or empty.

(E8.5) [Direct Products]

Let $\mathbb{F}$ be a field, V an $\mathbb{F}$-vector space and $U, W \subseteq V$ subspaces of V, with $V = U \oplus W$.

(i) Show that $\begin{aligned} \varphi : V &\longrightarrow V/U \\ \mathbf{v} &\longmapsto \mathbf{v} + U \end{aligned}$ is a linear map.

(ii) Determine $\ker(\varphi)$.

(iii) Show that $\begin{aligned} \psi : V &\longrightarrow V/U \times V/W \\ \mathbf{v} &\longmapsto (\mathbf{v} + U, \mathbf{v} + W) \end{aligned}$ is an isomorphism.