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7th Exercise Sheet Linear Algebra I for MCS
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(E7.1) [Subspaces]

Suppose V is an n-dimensional vector space over a field $\mathbb{F}$. Let W be a subspace of V with $\dim(W) = r < n$. Show that W is the intersection of all $(n-1)$-dimensional subspaces U of V with $W \subseteq U$.

(E7.2) [Linear (in)dependence]

Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ be linearly dependent vectors of an $\mathbb{F}$-vector space V, such that any three of them are linearly independent.

(i) Find an example of such vectors in $\mathbb{R}^3$ over $\mathbb{R}$.

(ii) Show: There are non-zero scalars λ_1, λ_2, λ_3 and λ_4 in $\mathbb{F}$ such that $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 + \lambda_4\mathbf{v}_4 = \mathbf{0}$. Moreover show that any other $(\mu_1, \mu_2, \mu_3, \mu_4) \in \mathbb{F}^4$ such that $\mu_1\mathbf{v}_1 + \mu_2\mathbf{v}_2 + \mu_3\mathbf{v}_3 + \mu_4\mathbf{v}_4 = \mathbf{0}$ is of the form $(\mu_1, \mu_2, \mu_3, \mu_4) = \nu(\lambda_1, \ldots, \lambda_4)$ for some $\nu \in \mathbb{F}$.

(E7.3) [Linear independence and dimension of subspaces in $\mathbb{R}^5$]

Consider the vectors

$$\mathbf{u}_1 = (0,1,0,1,0), \quad \mathbf{u}_2 = (1,0,0,0,0), \quad \mathbf{u}_3 = (1,0,1,0,1),$$
$$\mathbf{w}_1 = (1,1,0,0,0), \quad \mathbf{w}_2 = (1,2,0,1,0), \quad \text{and} \quad \mathbf{w}_3 = (1,1,1,0,1).$$

Let U be the subspace of $\mathbb{R}^5$ spanned by $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ and W be the subspace spanned by $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$.

(i) Select bases for U and W from $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, respectively.

(ii) What are the dimensions of U and W?

(iii) Determine $U \cap W$.

(iv) Find a basis of $U \cap W$.

(v) Extend the basis from (iv) to bases of U and W in such a way that you will get a basis of $U + W = \mathrm{span}(U \cup W)$ as well. What is the dimension of $U + W$?

(E7.4) [**Direct sums**]

Consider the following subsets of the $\mathbb{R}$-vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$:
$V_1 := \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) : f(x) = f(-x) \text{ for all } x \in \mathbb{R}\}$, the subset of even functions;
$V_2 := \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) : -f(x) = f(-x) \text{ for all } x \in \mathbb{R}\}$, the subset of odd functions.

(i) Show that V_1, V_2 are subspaces of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

(ii) Show that $\mathcal{F}(\mathbb{R}, \mathbb{R}) = V_1 \oplus V_2$.

(E7.5) [**Cardinalities of finite fields**]

The aim of this exercise is to find the possible cardinalities for finite fields.

(i) Warm up: Prove (without referring to Theorem 2.5.18 of the notes) that every *finite* $\mathbb{F}$-vector space V is isomorphic to $\mathbb{F}^n$ for some $n \in \mathbb{N}$.

Let $\mathbb{F}$ be a finite field. Define for each $n \in \mathbb{N}$ the element $\underline{n} \in \mathbb{F}$: $\underline{0} := 0$ and $\underline{n+1} := \underline{n} + 1$ such that $\underline{n} = \underbrace{1 + \cdots + 1}_{n \text{ times}}$. Put $F := \{\underline{n} : n \in \mathbb{N}\}$ and let $p := |F|$.

(ii) Prove that the set F is a subring of $\mathbb{F}$ that is isomorphic to the ring $\mathbb{Z}_p$.

(iii) Prove that p is a prime[1] and conclude that F is a field.

(iv) Use E4.2 and (i) to find all possible cardinalities of $\mathbb{F}$.

Remark: One can also show that for each of these possible cardinalities there exists (up to isomorphisms) exactly one field, but this is beyond the scope of linear algebra. If you want you can try to explicitly construct a (the) field with four elements.

[1] This number p is called the *characteristic* of $\mathbb{F}$.