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6th Exercise Sheet Linear Algebra I for MCS
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(E6.1) [Linear independence for vectors in $\mathbb{F}^5$]

Consider $\mathbf{v}_1 = (1,1,0,1,1), \mathbf{v}_2 = (0,0,1,1,0), \mathbf{v}_3 = (0,1,0,0,0), \mathbf{v}_4 = (1,0,1,0,1), \mathbf{v}_5 = (1,0,0,1,0)$ as elements of $\mathbb{F}^5$, where

(i) $\mathbb{F} = \mathbb{R}$,

(ii) $\mathbb{F} = \mathbb{Z}_2$.

In each case determine whether the vectors are linearly dependent or linearly independent. If the vectors are linearly dependent then determine a maximal linearly independent subset.

(E6.2) [Systems of linear equations]

Consider the following system of linear equations over $\mathbb{R}$:

$$\begin{array}{ccccccccccc} x & + & 2y & + & 2z & - & s & + & 3t & = & 0 \\ x & + & 2y & + & 3z & + & s & + & t & = & 0 \\ 3x & + & 6y & + & 8z & - & s & + & 5t & = & 0 \end{array} .$$

Find the dimension and a basis of the solution space W of this system over $\mathbb{R}$.

(E6.3) [Linear independence under isomorphisms]

Let $\mathbb{F}$ be a field, V and W $\mathbb{F}$-vector spaces, $\varphi : V \to W$ an $\mathbb{F}$-vector space isomorphism, and $S \subseteq V$. Prove that S is linearly independent if, and only if, $\varphi(S)$ is linearly independent.

(E6.4) [Bases of finite vector spaces]

(i) Write down all possible bases of $\mathbb{Z}_2^2$.

(ii) How many different bases does $\mathbb{Z}_2^3$ have?

(iii) Give a general formula for the n-dimensional case $\mathbb{Z}_2^n$.

(iv) Give a general formula for the number of bases of $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is a field with q elements.

(E6.5) [Fibonacci sequences]

Consider the $\mathbb{R}$-vector space Fib of all Fibonacci sequences (compare Example 2.2.4 in the lecture notes), where Fib was considered as a subspace of $\mathcal{F}(\mathbb{N}, \mathbb{R})$).

Recall that a sequence $(a_i)_{i\in\mathbb{N}} = (a_0, a_1, a_2, \ldots)$ is a Fibonacci sequence if:

$$a_{i+2} = a_i + a_{i+1} \text{ for all } i \in \mathbb{N}.$$

(i) Show that the space Fib has a basis consisting of two sequences; hence its dimension is 2. (Note that every Fibonacci sequence is determined by its first two members.)

(ii) Show that Fib contains precisely two different geometric sequences, i.e., sequences of the form

$$a_i = \rho^i, \quad i = 0, 1, \ldots$$

for some real $\rho \neq 0$. (Hint: $\rho_1 = \frac{1+\sqrt{5}}{2}$, $\rho_2 = \frac{1-\sqrt{5}}{2}$.)

(iii) Show that the two sequences from part (ii) are linearly independent, hence form a basis.

(iv) Express the standard Fibonacci sequence $\mathbf{f} = (0, 1, 1, 2, 3, \ldots)$ as a linear combination in the basis obtained in part (iii) and give a closed term representation for the i-th member in $\mathbf{f}$.