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5th Exercise Sheet Linear Algebra I for MCS
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(E5.1) [Subspaces]

Which of the following subsets of vector spaces are affine subspaces? Which of them are linear subspaces?

(i) $\{(1,1)\}$ in the vector space $\mathbb{R}^2$ over $\mathbb{R}$,

(ii) $\{(a,b) \in \mathbb{R}^2 : \ a = 0\}$ in the vector space $\mathbb{R}^2$ over $\mathbb{R}$,

(iii) $\{(a,b,c) \in \mathbb{R}^3 : \ a+b+c = 1\}$ in the vector space $\mathbb{R}^3$ over $\mathbb{R}$,

(iv) $\{(a,b,c) \in \mathbb{R}^3 : \ abc = 1\}$ in the vector space $\mathbb{R}^3$ over $\mathbb{R}$,

(v) $\{f \in \mathcal{F}(\mathbb{R},\mathbb{R}) : \ f(1) = 1\}$ in the vector space $\mathcal{F}(\mathbb{R},\mathbb{R})$ over $\mathbb{R}$,

(vi) $\{x \in \mathbb{R} : x + \sqrt{2} \in \mathbb{Q}\}$ in the vector space $\mathbb{R}$ over $\mathbb{Q}$.

(E5.2) [Combinations of subspaces]

Let U and W be two linear subspaces of $\mathbb{R}^3$ with

$$U = \{\lambda \cdot (1,1,0) : \lambda \in \mathbb{R}\} \quad \text{and} \quad W = \{\mu \cdot (-1,1,0) : \mu \in \mathbb{R}\}$$

(i) Draw U and W.

(ii) Draw $U \cap W$, $U \cup W$ and $U + W := \{u + w : u \in U, w \in W\}$. Which of these are a linear subspaces?

(E5.3) [Intersection of affine subspaces]

Let $\mathbb{F}$ a field and V an $\mathbb{F}$-vector space.

(i) Let $\mathbf{v}, \mathbf{w} \in V$ and $U_i \subseteq V, i = 1,2$, be subspaces. Determine $\mathbf{v} + U_1 \cap \mathbf{w} + U_2$.

(ii) Now in general, show that the intersection of any number of affine subspaces of a vector space V is either empty or again an affine subspace.

(E5.4) [Dedekind identity]

Let $\mathbb{F}$ a field, V an $\mathbb{F}$-vector space and $U_i \subseteq V,\ i = 1, 2, 3,$ be subspaces. Moreover, let $U_1 \subseteq U_3$. Show the following:

$$\operatorname{span}(U_1 \cup U_2) \cap U_3 = \operatorname{span}(U_1 \cup (U_2 \cap U_3))$$

(E5.5) [A game of affine subspaces]

Consider the three-element field $\mathbb{F}_3$ and associated vector spaces $\mathbb{F}_3^n$.

(i) Show that in $\mathbb{F}_3$, the solutions to the linear equation $x+y+z=0$ are precisely those triples $(x, y, z) \in \mathbb{F}_3^3$ in which all components are equal or all components distinct.

(ii) Show that a set of three points $\mathbf{v}_1 = (v_{11}, \ldots, v_{1n}), \mathbf{v}_2 = (v_{21}, \ldots, v_{2n}), \mathbf{v}_3 = (v_{31}, \ldots, v_{3n})$ in $\mathbb{F}_3^n$ forms an affine subspace of $\mathbb{F}_3^n$ if, and only if, for every $i \in \{1, \ldots, n\}$ either $v_{1i} = v_{2i} = v_{3i}$ or $\{v_{1i}, v_{2i}, v_{3i}\} = \mathbb{F}_3$.

The game SET has 81 cards which differ according to four properties for each of which there are three distinct states

colour :	red, green, blue;
shape :	round, angular, wavy;
numbers :	1, 2, 3;
face :	thin, medium, bold.

A "*set*", in terms of the rules of the game, is any set of three cards such that for each one of the four properties, either all three states of that property are represented or all three cards in the set have the same state.[1]

(iii) Give a correspondence between the cards in SET and $\mathbb{F}_3^4$ such that a "*set*" is the same as a 3-element affine subspace (a line).

(iv) Determine the number of "*sets*", "*sets*" containing a given card, and "*sets*" containing two distinct given cards.

(v) Varying the number of properties (and cards), consider SET^n for $n \geqslant 2$, with 3^n cards. Show that there is a collection of 2^n cards that does not contain any "*set*". Can you find an even larger such collection in the case of $n = 4$?

(∗) A problem for further speculation: what is the maximal size of a "*setless*" subset of points in SET^n? Are there any better general lower bounds than the one from (v)? (Note that we do not know the answers either).

[1] You can play this game also online at `http://setgame.freehostia.com/`.