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4th Exercise Sheet Linear Algebra I for MCS
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(E4.1) [Subgroups]

A subset $H \subseteq G$ of a group $(G, *, e)$ forms a *subgroup* of G if H is a group w.r.t. the restriction of the operation $*$ to $H \times H$. (In fact this is equivalent to $h_1 * h_2 \in H$ and $h_1^{-1} \in H$ for all $h_1, h_2 \in H$.)

Consider symmetric groups $(\mathrm{Sym}(A), \circ, \mathrm{id}_A)$ and in particular $S_3 = (\mathrm{Sym}(\{1,2,3\}), \circ, \mathrm{id})$ and $S_4 = (\mathrm{Sym}(\{1,2,3,4\}), \circ, \mathrm{id})$.

(i) Show that for every $a \in A$, the set of permutations of A that fix a, $H_a = \{f \in \mathrm{Sym}(A)\colon f(a) = a\}$ forms a subgroup of $(\mathrm{Sym}(A), \circ, \mathrm{id}_A)$.

(ii) Show that S_3 is isomorphic to a subgroup of S_4.

(iii) Find all the subgroups of S_3. (Hint: There are six.)

(iv) Find a proper subgroup of S_4 that is not isomorphic to S_3 or one of its subgroups.

(E4.2) [Subspaces]

Determine whether or not W is a subspace of the $\mathbb{R}$-vector space V.

(i) Let $V = \mathbb{R}^3$.

(a) $W = \{(a, b, 0) : a, b \in \mathbb{R}\}$

(b) $W = \{(a, b, c) : a^2 + b^2 + c^2 \leq 1, a, b, c \in \mathbb{R}\}$

(c) $W = \{(a, b, c) : a, b, c \in \mathbb{Q}\}$

(ii) Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$.

(a) $W = \{f : f(3) = f(7)\}$

(b) W consists of the odd functions, i.e., those functions f for which $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

(c) $W = \{f : f(7) = f(1) + 2\}$

(E4.3) [Lines]

Let $\mathbb{F}$ be a filed, V an $\mathbb{F}$-vector space. Affine subspaces of the form $S_{\mathbf{u},\mathbf{v}} := \{\mathbf{u}+\lambda\mathbf{v} : \lambda \in \mathbb{F}\}$ for some $\mathbf{u}, \mathbf{v} \in V, \mathbf{v} \neq \mathbf{0}$ are called *lines*.

(i) Show that for any two distinct $\mathbf{u}_1, \mathbf{u}_2 \in V$ there is a unique line containing $\mathbf{u}_1$ and $\mathbf{u}_2$. When is this affine subspace a linear subspace?

(ii) Consider the family of all lines $S_{\mathbf{u},\mathbf{v}}$, for a fixed $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ and all $\mathbf{u} \in V$.

  (a) Show that two such lines are either identical or disjoint.

  (b) Check that "belonging to the same line" from this family is an equivalence relation on V. When are two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ equivalent in this sense?

(iii) For $V = \mathbb{F}^n$, $\mathbb{F}$ a finite field of q elements:

  (a) How many points are on a line?

  (b) Of how many different lines does the family in (ii) consist?

(E4.4) [Subfields and vector spaces]

A subset $F \subseteq \mathbb{F}$ of a field $(\mathbb{F}, +, \cdot, 0, 1)$ forms a *subfield* of $\mathbb{F}$ if F is a field w.r.t. the restriction of the operations $+$ and $\cdot$ to $F \times F$ (it therefore needs to comprise 0 and 1).

Let $\mathbb{F}$ a field and F a subfield of $\mathbb{F}$. Prove that $\mathbb{F}$ is a vector space over F (with the natural choices for vector addition and scalar multiplication).

(In particular, we may regard, for instance, $\mathbb{R}$ as a $\mathbb{Q}$-vector space, or $\mathbb{C}$ is an $\mathbb{R}$-vector space.)