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3rd Exercise Sheet Linear Algebra I for MCS
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(E3.1) [Isomorphisms]

Let $(A, *^A, e^A)$ and $(B, *^B, e^B)$ two structures with a binary operation and a distinguished element. Let $\varphi : A \to B$ be a bijective map such that

$$\begin{array}{ll} \text{for all } x, y \in A: & \varphi(x *^A y) = \varphi(x) *^B \varphi(y) \\ \text{and} & \varphi(e^A) = e^B. \end{array}$$

Prove that $(A, *^A, e^A)$ is a group if and only if $(B, *^B, e^B)$ is a group.

(E3.2) [Groups and finite fields]

(i) Let p be a prime. Show that for any integer a not divisible by p there is an integer k such that $ka = 1 \bmod p$. (Hint: recall from the OWO lecture that $\gcd(s, t)$ can be expressed as an integer linear combination of s and t.)

(ii) Let p again be prime. Show that in case p divides the product ab of two integers a and b, it must divide a or b.

(iii) Let n be a positive integer. Show that $(\mathbb{Z}_n \setminus \{0\}, \cdot_n, 1)$ is a group iff n is prime.

(E3.3) [Group theory]

(i) Which of the following groups are isomorphic?

1) $(\mathbb{Z}_2^2, +, \mathbf{0})$
2) $(\mathbb{Z}_4, +, 0)$
3) $(\mathbb{Z}_5 \setminus \{0\}, \cdot, 1)$
4) $(\{x \in \mathbb{C} : x^4 = 1\}, \cdot, 1)$
5) $(\{x \in \mathbb{Z}_8 : 4x = 0\}, +, 0)$

(ii) Find (up to isomorphism) all groups that have exactly 4 elements.

(Hint: Analyse the possible choices by looking at the tables for the group operation and making suitable case distinctions when there are choices.)

(E3.4) [Rings and fields]

Let $(R, +, \cdot, 0, 1)$ be a ring. Prove that

(i) $0 = 1$ if and only if $R = \{0\}$,

(ii) if R is a field then $a \cdot b = 0$ implies $a = 0$ or $b = 0$[1].

(E3.5) [Abelian groups and isomorphisms]

Let $(G, *, e)$ a group and $\alpha : G \to G, g \mapsto g'$, where $g * g' = e$. Show that α is an isomorphism if and only if G is abelian.

[1]This shows that a field has now *zero divisors* and is a so-called *integral domain*.