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2. Exercise Sheet Linear Algebra I for MCS
Winter Term 2006/2007

Solutions of exercises E2.2-E2.5 to be handed in in the lecture on ***Tuesday, October 31****.*

(E2.1) [Sets and functions]

Let A, B, M, N be sets and $f : M \to N, g : N \to M$ functions.

(i) (De Morgan Laws) Show that

$$M\backslash(A \cup B) = (M\backslash A) \cap (M\backslash B) \qquad \text{and} \qquad M\backslash(A \cap B) = (M\backslash A) \cup (M\backslash B).$$

(ii) Let $(A\backslash B) \cup (B\backslash A) = A \cup B$. Show that then $A \cap B = \emptyset$.

(iii) Let $A, B \subseteq M$. Show that

$$f(A \cup B) = f(A) \cup f(B) \qquad \text{and} \qquad f(A \cap B) \subseteq f(A) \cap f(B).$$

Find an example where $f(A \cap B) \neq f(A) \cap f(B)$.

(iv) Suppose $f \circ g = id_N$. What does this imply about injectivity/surjectivity of the functions f and g?

(E2.2) [Systems of linear equations]

Consider the following system of linear equations:

$$E : \left\{ \begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 3 \\ x & + & 3y & + & 5z & = & 4 \\ 3x & + & 7y & + & 11z & = & 10 \end{array} \right.$$

(i) Use the Gauß-Jordan method to determine the set of solutions $S(E^*)$ for the homogeneous system E^*.

(ii) Verify in this particular instance that $\mathbf{0} \in S(E^*)$ and that $S(E^*)$ is closed under vector addition and scalar multiplication (cf. Lemma 1.1.4 (i)).

(iii) Use one selected solution $\mathbf{v}_0 \in S(E)$ of the inhomogeneous system E to obtain $S(E)$ according to Lemma 1.1.4 (ii).

(iv) Compare the result from (ii) with what you get from a direct application of Gauß-Jordan to E rather than E^*.

(E2.3) [Systems of linear equations with parameters]

Determine all real numbers $a \in \mathbb{R}$ such that the following system of linear equations is solvable and compute the corresponding solution set:

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & + & 2x_3 & = & 2 \\ 2x_1 & + & a^2 x_2 & + & 2x_3 & = & a \\ 2x_1 & + & x_2 & + & x_3 & = & 1 \end{array}$$

(E2.4) [Systems of linear equations over $\mathbb{Z}_2$]

Consider the following system of linear equations over $\mathbb{Z}_2$ (the field $\mathbb{F}_2 = (\mathbb{Z}_2, +_2, \cdot_2, 0, 1)$ of two elements):

$$E \left\{ \begin{array}{lrrrrcl} E_1: & x_1 & & & + \; x_4 & = & 0 \\ E_2: & x_1 & + \; x_2 & & + \; x_4 & = & 1 \\ E_3: & & + \; x_2 & + \; x_3 & + \; x_4 & = & 0 \\ E_4: & x_1 & + \; x_2 & + \; x_3 & & = & 1 \end{array} \right.$$

(i) List (in a table) all elements of $\mathbb{Z}_2^4$ and determine the solution set of each individual equation E_j and of the whole system E "by hand", by going through the table and checking for each entry which of the individual equations are satisfied.

(ii) Similarly consider the associated homogeneous system.

(iii) Determine the solution set of the homogeneous system and of the inhomogeneous system using the ($\mathbb{Z}_2$-analogue of the) Gauß-Jordan procedure.

(E2.5) [Permutations]

Let $f, g \in S_4$, where $f = \binom{1\,2\,3\,4}{4\,3\,2\,1}$ and $g = \binom{1\,2\,3\,4}{4\,1\,2\,3}$.

(i) How many elements does S_4 have? Compute f^{-1}, $f \circ g$ and $g \circ f$.

(ii) Let $g^1 := g$ and $g^n := g \circ g^{n-1}$ for $n \geqslant 1$. Compute g^{777}.