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14th Exercise Sheet Linear Algebra II for MCS
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(E14.1) [Geometric properties of the ellipse in euclidean geometry]

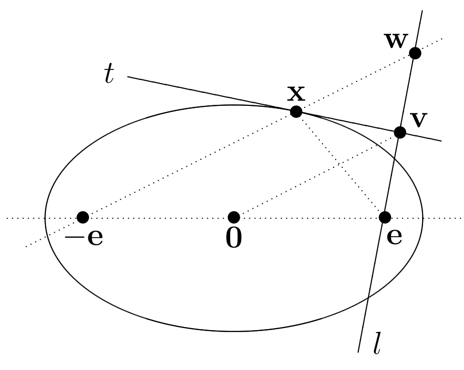
Let $0 \leqslant e < 1$ and consider the points given by the vectors $\mathbf{e} = (e, 0)$ and $-\mathbf{e} = (-e, 0)$ in the real plane $\mathbb{R}^2$. Let the set $\mathbb{X}_e \subseteq \mathbb{R}^2$ be defined as the set of all those $\mathbf{x} \in \mathbb{R}^2$ for which $d(\mathbf{x}, \mathbf{e}) + d(\mathbf{x}, -\mathbf{e}) = 2$.

(i) Show that $\mathbb{X}_e$ is an *ellipse* defined by a quadratic equation of the form $\alpha x^2 + \beta y^2 = 1$ for suitable $\alpha, \beta > 0$. Determine α and β in terms of e. Draw $\mathbb{X}_e$ for $e = 0, 1/2, 1, 1/\sqrt{2}$.

(ii) From (i) find a representation of $\mathbb{X}_e$ as the image of the unit circle under a rescaling in the y-direction. Use this rescaling and the fact that linear transformations preserve the property that a line is a tangent to a curve in order to determine the equation of the tangent to the ellipse $\mathbb{X}_e$ in a point $\mathbf{x} = (x, y) \in \mathbb{X}_e$. Show that lines from $\mathbf{e}$ and $-\mathbf{e}$ through $\mathbf{x}$ form the same angle with the tangent at $\mathbf{x}$. [This explains the rôle of the points $\mathbf{e}$ and $-\mathbf{e}$ as the *foci* of the ellipse: light shining from $\mathbf{e}$ is focussed in $-\mathbf{e}$ after reflection in $\mathbb{X}_e$.]

(iii) Show by elementary geometric means that $\mathbb{X}_e$ also has the following geometric property. Let t be the tangent to $\mathbb{X}_e$ in a point $\mathbf{x} \in \mathbb{X}_e$ and l the line through $\mathbf{e}$ perpendicular to t. Then the point of intersection $\mathbf{v}$ between l and t lies on the unit circle.

Hint: look at the triangles $(\mathbf{x}, \mathbf{v}, \mathbf{w})$ and $(\mathbf{x}, \mathbf{v}, \mathbf{e})$ in the sketch below, where $\mathbf{v}$ marks the point where l intersects t, and $\mathbf{w}$ where it intersects the line through $-\mathbf{e}$ and $\mathbf{x}$. Use part (ii) to argue that these triangles are congruent.



(E14.2) [Extra: bedding a snake]

A snake of length 1 (yard, meter, ...) wants to buy a blanket that is guaranteed to cover it no matter which shape it finds comfortable for resting. Obviously a circular blanket of diameter 1 would do (why?). A clever shop assistant recommends to buy just half that, a blanket of the shape of a half disc of diameter 1. Is that safe to buy?

Mathematically: show that any (rectifiable, smooth, ..., largely you can put what you like) plane curve of unit length in $\mathbb{R}^2$ can be covered by a half disc congruent to $\{(x, y) : y \geqslant 0, x^2 + y^2 \leqslant 1\}$.

Hint: consider the mid-point of the snake and find two suitable ellipses that cover it; then argue that the union of the ellipses is covered by a half disc as required. [The solution to this exercise will only be revealed to serious contenders!]