Fachbereich Mathematik
Martin Otto
Benno van den Berg

TECHNISCHE UNIVERSITÄT DARMSTADT

July 12, 2007

13th Exercise Sheet Linear Algebra II for MCS
Summer Term 2007

(E13.1) [Quadratic forms]

Consider the quadratic form $Q : \mathbb{R}^2 \to \mathbb{R}$ given by

$$Q(x_1, x_2) = \frac{1}{10}(21x_1^2 + 29x_2^2 - 6x_1x_2).$$

(i) Find a symmetric bilinear form $\sigma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ with $\sigma(\mathbf{x}, \mathbf{x}) = Q(\mathbf{x})$.

(ii) Determine an orthogonal basis for which σ is described by a diagonal matrix.

(E13.2) [Quadratic forms]

Find the principal axes of the quadratic forms

(i) $Q_1(\mathbf{x}) = 5x_1^2 - 4x_1x_2 + 8x_2^2$,

(ii) $Q_2(\mathbf{x}) = 6x_1^2 - 20x_1x_2 + 6x_2^2$,

(iii) $Q_3(\mathbf{x}) = \frac{2}{5}(x_1^2 - 6x_1x_2 + 9x_2^2)$,

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and determine their signature.

(E13.3) [Quadrics]

Consider the quadric $\mathbb{X}_{\lambda,\mu}$ in $\mathbb{R}^3$ defined by

$$\mathbb{X}_{\lambda,\mu} := \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \lambda(x_1^2 + x_2^2) + \mu x_3^2 = 1\},$$

where λ and μ are real parameters.

(i) Determine the intersection of every $\mathbb{X}_{\lambda,\mu}$ with the plane defined by $x_3 = c \in \mathbb{R}$.

(ii) Prove that $\mathbb{X}_{\lambda,\mu}$ can be obtained by rotating the set

$$\mathbb{X}'_{\lambda,\mu} := \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 0, \lambda x_2^2 + \mu x_3^2 = 1\}$$

about the x_3-axis.

(iii) For each pair of values

	λ	μ
1.	-1	1
2.	1	-1
3.	2	1,

sketch $\mathbb{X}_{\lambda,\mu}$ and $\mathbb{X}'_{\lambda,\mu}$.

(E13.4) [Bilinear and quadratic forms]

(i) Let the bilinear form $\sigma : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be given by $\sigma(\mathbf{x}, \mathbf{y}) = \mathbf{x}^t A \mathbf{y}$, where

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 3 & 1 & 4 \\ 5 & 4 & 9 \end{pmatrix}.$$

Are there any (non-trivial) subspaces $U \subseteq \mathbb{R}^3$ such that the linear maps

$$\begin{array}{rccc} \sigma_{\mathbf{u}} : & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & \mathbf{x} & \longmapsto & \sigma(\mathbf{u}, \mathbf{x}) \end{array}$$

are identically zero for all $\mathbf{u} \in U$? How is the existence of such a subspace related to the signature of σ?

Similarly investigate $\sigma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by $\sigma(\mathbf{x}, \mathbf{y}) = \mathbf{x}^t B \mathbf{y}$, with $B = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$.

(ii) In both cases, consider the associated quadratic forms $Q(\mathbf{x}) = \sigma(\mathbf{x}, \mathbf{x})$.
For which subspaces U are the restrictions of Q identically zero?

(E13.5) [Quadrics]

Consider the quadric

$$\mathbb{X} = \{\mathbf{v} \in \mathbb{R}^n : Q(\mathbf{v}) = c\},$$

where Q is a quadratic form over $\mathbb{R}^n, c \in \mathbb{R}$. Show that $\mathbb{X}$ is invariant under the following linear isometries of $\mathbb{R}^n$:

(i) $-\,\mathrm{id} : \mathbf{x} \mapsto -\mathbf{x}$ (central symmetry);

(ii) reflection in hyperplanes orthogonal to a principal axis (i.e., hyperplanes spanned by any $n-1$ basis vectors from an orthonormal basis that diagonalises Q and the associated σ);

(iii) rotations in planes spanned by two principal axes w.r.t. which Q has the same "eigenvalues", i.e., by basis vectors $\mathbf{b}, \mathbf{b}'$ from an orthonormal basis that diagonalises Q such that $Q(\mathbf{b}) = Q(\mathbf{b}')$.