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12th Exercise Sheet Linear Algebra II for MCS
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(E12.1) [Hermitian matrices]

Let $A \in \mathbb{C}^{(n,n)}$. Show that the following are equivalent:

(i) A is hermitian.

(ii) A is normal and all its eigenvalues are real numbers.

(iii) $\langle \mathbf{v}, A\mathbf{v} \rangle \in \mathbb{R}$ for all $\mathbf{v} \in \mathbb{C}^n$.

Hint: Consider $\langle \mathbf{v} + \mathbf{w}, A(\mathbf{v} + \mathbf{w}) \rangle$ and $\langle \mathbf{v} + i\mathbf{w}, A(\mathbf{v} + i\mathbf{w}) \rangle$ for the implication from (iii) to (i).

(E12.2) [Simultaneous diagonalisation]

Let φ and ψ be endomorphisms of a finite dimensional vector space V such that $\varphi \circ \psi = \psi \circ \varphi$ (φ and ψ commute).

(i) Show that any eigenspace of φ is invariant under ψ and vice versa.

(ii) Use part (i) to show that if φ and ψ both have a basis of eigenvectors, then they are *simultaneously diagonalisable* in the sense that there is a basis $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ of V such that every $\mathbf{b}_i$ is an eigenvector of both φ and ψ.

(E12.3) [Positive definiteness]

Let $A \in \mathbb{R}^{(n,n)}$ be symmetric. We write B^n_r for the ball of radius r in $\mathbb{R}^n$.
Show that the following are equivalent:

(i) A is positive definite.

(ii) There is some $R > 0$ such that $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^t A \mathbf{x} \leqslant 1\} \subseteq B^n_R$.

Show that for any scalar product σ on $\mathbb{R}^n$ there are numbers $0 < r < R$ such that

$$B^n_r \subseteq \{\mathbf{x} \in \mathbb{R}^n : \sigma(\mathbf{x}, \mathbf{x}) \leq 1\} \subseteq B^n_R.$$

This means that any two scalar products on R^n induce metrics that are *commensurate* (in particular they induce the same topology).

(E12.4) [Preservation of quadratic forms]

Let σ be a symmetric bilinear form on $\mathbb{R}^n$, represented by $A \in \mathbb{R}^{(n,n)}$ w.r.t. the standard basis, and let Q be the associated quadratic form $Q(\mathbf{x}) = \sigma(\mathbf{x}, \mathbf{x})$.

We say that an endomorphism φ of $\mathbb{R}^n$ preserves the bilinear from σ if $\sigma(\varphi(\mathbf{x}), \varphi(\mathbf{y})) = \sigma(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Analogously, φ preserves the associated quadratic form Q if $Q(\varphi(\mathbf{x})) = Q(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.

Show that for an endomorphism φ represented by the matrix C w.r.t. the standard basis, the following are equivalent:

(i) φ preserves Q;

(ii) φ preserves σ;

(iii) $C^t AC = A$.

(E12.5) [Transformation of quadratic forms]

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the endomorphism described, w.r.t. the standard basis of $\mathbb{R}^2$, by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The unit circle S^1 in $\mathbb{R}^2$ is the solution set to the equation $x_1^2 + x_2^2 = 1$,

$$S^1 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + x_2^2 = 1\}.$$

Describe the image of the unit circle under φ, $\varphi[S^1] \subseteq \mathbb{R}^2$, by a corresponding equation. Find a symmetric bilinear form σ for which $\varphi[S^1] = \{\mathbf{x} \in \mathbb{R}^2 \colon \sigma(\mathbf{x}, \mathbf{x}) = 1\}$. Use diagonalisation according to Theorem 3.2.5 to find the symmetry axes of $\varphi[S^1]$.