Fachbereich Mathematik
Martin Otto
Benno van den Berg

TECHNISCHE UNIVERSITÄT DARMSTADT

June 28, 2007

11th Exercise Sheet Linear Algebra II for MCS
Summer Term 2007

(E11.1) [Orientation preserving orthogonal maps]

(i) Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be an orientation preserving orthogonal map. Show that for any set of vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^n$:

$$\det(\varphi(\mathbf{a}_1), \varphi(\mathbf{a}_2), \dots, \varphi(\mathbf{a}_n)) = \det(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n).$$

(ii) Show that any orientation preserving orthogonal map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ preserves the cross product.

Hint: recall from Exercise (E14.2) from Linear Algebra I that the cross product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ is the unique vector in $\mathbf{a} \times \mathbf{b} \in \mathbb{R}^3$ such that $\langle \mathbf{a} \times \mathbf{b}, \mathbf{x} \rangle = \det(\mathbf{a}, \mathbf{b}, \mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$.

(iii) Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be the orientation preserving orthogonal map with $\varphi(2,1,2) = (0,3,0)$ and $\varphi(0,-3,0) = (2,-1,-2)$. Determine the matrix representation of φ with respect to the standard basis, and interpret φ geometrically.

Hint: use (ii).

(E11.2) [Orthogonal diagonalisability]

Find an *orthogonal* matrix C such that the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

is transformed into a diagonal matrix by $C^{-1}AC = C^tAC$. Which property of A guarantees that you can find such a C?

(E11.3) [Matrix groups]

(i) Show that the special orthogonal group $SO(n)$ is the subgroup of $O(n)$ consisting of those matrices that represent orientation preserving orthogonal maps in $\mathbb{R}^n$ w.r.t. the standard orthonormal basis. (Compare Exercise 2.3.11 on page 74 of the notes.)

(ii) Prove that $U(1)$ and $SO(2)$ are isomorphic as groups.

Hint: use that $\mathbb{C}\backslash\{0\}$ is isomorphic to a certain subgroup of $\mathrm{GL}_2(\mathbb{R})$, as in Exercise (T5.2) from Linear Algebra I.

(E11.4) [Projections]

Let V be a finite dimensional euclidean or unitary vector space. Suppose π is an endomorphism of V that is a projection, i.e., $\pi \circ \pi = \pi$.

Show that π is self-adjoint if, and only if, π is an orthogonal projection.

(E11.5) [Normal matrices]

Let $A \in \mathbb{C}^{(n,n)}$ be normal. Show that if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then it is also an eigenvalue of A^+ with eigenvalue $\bar{\lambda}$.

Hint: consider $\langle A^+\mathbf{v} - \bar{\lambda}\mathbf{v}, A^+\mathbf{v} - \bar{\lambda}\mathbf{v}\rangle$.