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10th Exercise Sheet Linear Algebra II for MCS
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(E10.1) [Orthogonal maps]

Set

$$A := \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Determine an orthogonal matrix P, for which $P^t AP$ is diagonal and compute $P^t AP$.

(E10.2) [Reflections]

Let V be a finite dimensional (or unitary) vector space. Let $\mathbf{a} \in V$ be a unit vector (i.e. $\|\mathbf{a}\| = 1$) and let the map $\sigma : V \to V$ be defined by $\sigma(\mathbf{x}) := \mathbf{x} - 2\langle \mathbf{a}, \mathbf{x}\rangle \mathbf{a}$.

(i) Show that σ is an orthogonal (respectively unitary) map.

(ii) Prove $\sigma \circ \sigma = \mathrm{id}_V$.

(iii) Verify that $\sigma(\mathbf{a}) = -\mathbf{a}$ and $\sigma(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \{\mathbf{a}\}^\perp = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{a}\rangle = 0\}$.

(iv) Interpret the map σ geometrically for $V = \mathbb{R}^3$ w.r.t. the standard scalar product. Which endomorphisms of $\mathbb{R}^3$ do we get in this manner for arbitrary other scalar products in $\mathbb{R}^3$?

(E10.3) [Orthogonality]

(i) Show that a matrix $A \in \mathbb{R}^{(n,n)}$ is orthogonal iff its column vectors form an orthonormal basis w.r.t. the standard scalar product, iff its row vectors form an orthonormal basis w.r.t. the standard scalar product.

(ii) Show that $\det(A) = \pm 1$ for any orthogonal matrix $A \in \mathbb{R}^{(n,n)}$.

(iii) Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by

$$\begin{pmatrix} 5 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 8 \\ -1 \end{pmatrix} \mapsto \begin{pmatrix} 4 \\ 7 \end{pmatrix}.$$

Is φ orthogonal? Compute the matrix of φ with respect to the standard basis. Give a geometric description of the map.

(E10.4) [Orthogonal maps]

(i) Show that an orthogonal map in $\mathbb{R}^2$ is either the identity, the reflection in the origin, a reflection in a line or a rotation (the first two being special cases of the fourth). Conclude that every orthogonal map in $\mathbb{R}^2$ is the composition of at most two reflections in a line.

(ii) Show that an orthogonal map in $\mathbb{R}^3$ is either the identity, a reflection in a plane, a reflection in a line, the reflection in the origin, a rotation about an axis or a rotation about an axis followed by a reflection in the plane orthogonal to the axis (the first four being special cases of the last two). Conclude that every orthogonal map in $\mathbb{R}^3$ is the composition of at most three reflections in a plane.
Extra: how about orthogonal maps in $\mathbb{R}^n$?

(E10.5) [Dual spaces]

Recall that for any $\mathbb{F}$-vector space V, the set $\mathrm{Hom}(V, \mathbb{F})$ of linear maps $V \to \mathbb{F}$ has again the structure of a vector space, with vector addition and scalar multiplication being defined pointwise, turning it into what is called the dual space of V (see Section 3.2.2 on page 87 of the notes of Linear Algebra I).

If V is a euclidean vector space, we have a map $\varphi_V : V \to \mathrm{Hom}(V, \mathbb{R})$ with $\varphi_V(\mathbf{w}) \in \mathrm{Hom}(V, \mathbb{R})$ for any $\mathbf{w} \in V$ defined by

$$\varphi_V(\mathbf{w})(\mathbf{v}) = \langle \mathbf{w}, \mathbf{v} \rangle, \text{ for all } \mathbf{v} \in V.$$

The aim of this exercise is to show that φ_V is an isomorphism if V is finite dimensional, but not necessarily if V is infinite dimensional.

(i) Show that φ_V is an injective linear map.

(ii) Show that φ_V is an isomorphism if V is finite dimensional.

From now on, let $W := \mathcal{F}(\mathbb{N}, \mathbb{R})$ and $V = \{f \in W : f(n) = 0 \text{ for all but finitely many } n\}$.

(iii) Check that V is a subspace of W.

(iv) Show that $\langle f, g\rangle = \sum_{n\in\mathbb{N}} f(n)g(n)$ defines a scalar product on V, turning $(V, \langle . , . \rangle)$ into a euclidean space. Check that $\langle f, g\rangle$ is defined if $f \in W$ and $g \in V$, but not necessarily if f and g both belong to W.

(v) Show that the map $\psi : W \to \mathrm{Hom}(V, \mathbb{R})$ with $\psi(f) \in \mathrm{Hom}(V, \mathbb{R})$ for any $f \in W$ defined by $\psi(f)(g) = \langle f, g\rangle$ is an isomorphism of vector spaces. Conclude from this that φ_V, which is ψ restricted to V, is not.
Hint: use that the functions $b_n \in V$ $(n \in \mathbb{N})$ defined by $b_n(i) = 1$ if $n = i$ and 0 otherwise, form an orthonormal basis for V.