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9th Exercise Sheet Linear Algebra II for MCS
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(E9.1) [Orthogonal complements]

(Exercise 2.3.5 on page 68 of the notes.) Let V be a euclidean or unitary vector space of finite dimension. Moreover, let U, U_1, U_2 be subspaces of V. Show that we have

(i) $(U^\perp)^\perp = U$.

(ii) $(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp$.

(iii) $(U_1 \cap U_2)^\perp = U_1^\perp + U_2^\perp$.

(E9.2) [Orthogonal projections]

(Exercise 2.3.2 on page 68 of the notes.) Let φ be an endomorphism of a finite dimensional euclidean or unitary vector space $(V, \langle . , . \rangle)$.

Show the equivalence of the following:

(i) φ is an orthogonal projection.

(ii) $\varphi \circ \varphi = \varphi$ and $\ker(\varphi) \perp \mathrm{image}(\varphi)$.

(iii) $\varphi \circ \varphi = \varphi$ and $\mathbf{v} - \varphi(\mathbf{v}) \perp \varphi(\mathbf{v})$ for all $\mathbf{v} \in V$.

(iv) $\mathbf{v} - \varphi(\mathbf{v}) \perp \mathrm{image}(\varphi)$ for all $\mathbf{v} \in V$.

(E9.3) [Orthogonal projections]

(Exercise 2.3.3 on page 68 of the notes.) Show that the orthogonal projections of an n-dimensional euclidean or unitary vector space V are precisely those endomorphisms φ of V that are represented w.r.t. a suitable orthonormal basis by a diagonal matrix with ones and zeroes on the diagonal.

(E9.4) [Orthogonal projections]

(Exercise 2.3.4 on page 68 of the notes.) Let U and W be two subspaces of a finite dimensional euclidean or unitary vector space V, with orthogonal projections π_U and π_W onto U and W, respectively.

Prove that the following statements are equivalent:

(i) π_U and π_W commute.

(ii) $\pi_W \circ \pi_U = \pi_{U \cap W}$.

(iii) $\pi_W \circ \pi_U$ is an orthogonal projection.

(iv) $U = (U \cap W) \oplus (U \cap W^{\perp})$.

(v) $W = (U \cap W) \oplus (U^{\perp} \cap W)$.

(E9.5) [An orthonormal basis]

Let $V := \mathrm{Pol}_2(\mathbb{R})$ be the $\mathbb{R}$-vector space of all polynomial functions over $\mathbb{R}$ of degree at most 2. On this vector space

$$\langle p_1, p_2 \rangle := \int_{-1}^{1} p_1(x)\, p_2(x)\, \mathrm{d}x$$

defines a scalar product, turning $(V, \langle\, .\, ,\, .\, \rangle)$ into a euclidean space (see Section 2.2 on page 62 of the notes).

Determine an orthonormal basis of V.