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7th Exercise Sheet Linear Algebra II for MCS
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(E7.1) [Jordan normal form]

Let $\varphi : V \to V$ be an endomorphism of a finite dimensional $\mathbb{C}$-vector space V. Which of the following situations can occur? In case they can, give an example, in case they cannot, prove this.

(i) (a) V is 6-dimensional, the minimal polynomial of φ is $(X-2)^5$, and the eigenspace of 2 has dimension 3.

(b) V is 6-dimensional, the minimal polynomial of φ is $(X-2)(X-3)^2$, and the eigenspace of 2 has dimension 3.

(ii) (a) φ has minimal polynomial $(X-2)^4$ and there is a vector $\mathbf{v} \in V$ with height 3.

(b) φ has minimal polynomial $(X-2)^4$ and there is a vector $\mathbf{v} \in V$ with height 6.

(c) φ has minimal polynomial $(X-2)^4$, but no vector in V has height 3.

(iii) (a) φ has characteristic polynomial $(X-2)^6$ and $\varphi^2 - \varphi - \mathrm{id} = \mathbf{0}$.

(b) $\varphi^2 - \varphi - 2\,\mathrm{id} = \mathbf{0}$ and φ has eigenvalues that are not real.

(iv) (a) V has a φ-invariant subspace of dimension 5, 2 is the only eigenvalue of φ, but there is no $\mathbf{v} \in V$ with $\dim[\![\mathbf{v}]\!] = 5$.

(b) 2 is the only eigenvalue of φ, $V = [\![\mathbf{v}]\!] \oplus [\![\mathbf{w}]\!]$ with $\dim[\![\mathbf{v}]\!] = 5$, but the Jordan normal form for V contains no block of size 5.

(v) (a) V can be written as the direct sum of two φ-invariant subspaces of dimension 4, but there occur no Jordan blocks of size greater than 3 in the Jordan normal form for φ.

(b) V can be written as the direct sum of two φ-invariant subspaces of dimension 4, and in the Jordan normal form of φ there is a Jordan block of size 5.

(E7.2) [Scalar product]

Show that the map

$$\langle\cdot,\cdot\rangle : \mathbb{C}^{(n,n)} \times \mathbb{C}^{(n,n)} \to \mathbb{C}, (A, B) \mapsto \mathrm{tr}(A^+B)$$

defines a complex scalar product on $\mathbb{C}^{(n,n)}$, if $\mathrm{tr}(A)$ is the trace and A^+ the adjoint of the matrix $A \in \mathbb{C}^{(n,n)}$.

(E7.3) [Norm]

Let V be a euclidean vector space. Show that

(i) $\|\mathbf{u}\| = \|\mathbf{v}\|$ if, and only if, $\langle \mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle = 0$. (This is Thales' Theorem; why?)

(ii) $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if, and only if, $\langle \mathbf{u}, \mathbf{v}\rangle = 0$.

(E7.4) [Basis transformations for (semi-)bilinear forms]

Compare Exercise 2.1.3 on page 58 of the notes.

(i) Let σ be a bilinear form on an n-dimensional $\mathbb{R}$-vector space V, represented by the matrix A with respect to the basis $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$. If $B' = (\mathbf{b}'_1, \dots, \mathbf{b}'_n)$ is another basis for V, find an expression for the matrix $A' = [\![\sigma]\!]^{B'}$ in terms of A, the basis transformation matrices $C = [\![\mathrm{id}_V]\!]^{B}_{B'}$ and $C^{-1} = [\![\mathrm{id}_V]\!]^{B'}_{B}$ as well as their transposes as appropriate.

(ii) Similarly for a semi-bilinear form σ of an n-dimensional $\mathbb{C}$-vector space V: if σ is represented by A w.r.t. a basis B, what is its representation A' w.r.t. a basis B' in terms of A, the basis transformations matrices, as well as their adjoints?

(iii) Consider the following bilinear form $\langle \dots, \dots \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ on the vector space $\mathbb{R}^2$:

$$\left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle := 7v_1w_1 - 5v_1w_2 - 5v_2w_1 + 4v_2w_2.$$

What is its representation with respect to the standard basis? Then compute its representation with respect to the basis $(\mathbf{b}_1, \mathbf{b}_2) = (\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix})$ directly, as well as by using the formula obtained in part (i).

(iv) Is the bilinear form σ in part (iii) symmetric? Is it positive definite?

(E7.5) [Hermitian semi-bilinear forms]

Show that a semi-bilinear form σ on a finite dimensional $\mathbb{C}$-vector space V is hermitian iff the matrix $A = [\![\sigma]\!]^B$ that represents σ w.r.t. some (any) basis B is hermitian. (This is Exercise 2.1.2 on page 57 of the notes.)