

May 24, 2007

6th Exercise Sheet Linear Algebra II for MCS Summer Term 2007

(E6.1) [Jordan normal form]

Find a regular matrix R and a matrix J in Jordan normal form such that $A = RJR^{-1}$, for the matrix $A := \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & -1 & -3 & -2 \\ -2 & 3 & 5 & 2 \\ -1 & 2 & 2 & 3 \end{pmatrix}$, whose characteristic polynomial is $p_A = (2-X)^4$.

(E6.2) [Jordan normal form for describing processes]

Suppose that $\mathbf{s}_n \in \mathbb{R}^3$ describes the state of a 3-dimensional system at step $n \in \mathbb{N}$ (for example, the position of a particle in space). Suppose the evolution of the system from stage n to $n + 1$ is described by

$$\mathbf{s}_{n+1} = A\mathbf{s}_n, \quad \text{where } A = \begin{pmatrix} -4 & 2 & -1 \\ -4 & 3 & 0 \\ 14 & -5 & 5 \end{pmatrix}.$$

Use a transformation of the given A into Jordan normal form in order to get a feasible formula for \mathbf{s}_n , as a function of n and the initial state \mathbf{s}_0 . Compute \mathbf{s}_{100} for $\mathbf{s}_0 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$.

(E6.3) [Jordan normal form and transpose]

- (i) Show that if the $A, B \in \mathbb{F}^{(n,n)}$ are similar, then so are A^t and B^t .
- (ii) Let $A \in \mathbb{F}^{(n,n)}$ be a matrix in Jordan normal form. Show that A is similar to A^t . Deduce that over \mathbb{C} every square matrix is similar to its transpose.

(E6.4) [Jordan normal form]

- (i) Let φ be an endomorphism of a ten-dimensional \mathbb{F} -vector space V . W.r.t. basis $B = (\mathbf{b}_1, \dots, \mathbf{b}_{10})$ let φ be represented by a Jordan normal form matrix with three Jordan

blocks for the same eigenvalue $\lambda \in \mathbb{F}$, of sizes 2, 3 and 5. Let $\psi := \varphi - \lambda \text{id}$. Complete the following table:

i	1	2	3	4	5	6	7	8	9	10
$\dim(\ker \psi^i)$					10	10	10	10	10	10

In the notation of Lemma 1.6.4 of the notes: for which $\mathbf{v} \in V$ does $[\mathbf{v}]$ have maximal dimension? Split the basis B in a way to obtain bases for the two invariant subspaces $V = [\mathbf{v}] \oplus V'$ (as in Claim 1.6.5). If φ' is the restriction of φ to V' , what is the matrix representation of φ' with respect to this basis? If $\psi' = \varphi' - \lambda \text{id}$, how is the corresponding table for ψ' related to the above?

- (ii) Now, let φ be another endomorphism of V with characteristic polynomial $(\lambda - X)^{10}$. Suppose we have the following data for $\psi = \varphi - \lambda \text{id}$:

i	1	2	3	4	5	6	7	8	9	10
$\dim(\ker \psi^i)$	3	5	7	8	9	10	10	10	10	10

Determine the Jordan normal form representation of φ from this data (up to permutation of Jordan blocks).

- (iii) (extra) In general, let φ_0 and φ_1 be two endomorphisms of \mathbb{F} -vector spaces V_0 and V_1 of the same finite dimension, with the same characteristic polynomial that splits into linear factors. Suppose moreover that for each eigenvalue λ of φ_0 and φ_1 , the tables for $\psi_0 = \varphi_0 - \lambda \text{id}$ and $\psi_1 = \varphi_1 - \lambda \text{id}$ are the same.

Sketch a proof for the similarity of φ_0 and φ_1 adapting the argument for the existence and uniqueness of the Jordan normal form. How can this be used to give a “different” proof for the similarity of A and A^t for any matrix $A \in \mathbb{C}^{(n,n)}$?

(E6.5) [Jordan normal form and real matrices]

Let $A \in \mathbb{R}^{(n,n)}$ with $n = 2m$. Assume the characteristic polynomial of A is $p_A = p_0^m$, where $p_0 \in \mathbb{R}[X]$ is an irreducible polynomial of degree 2 in $\mathbb{R}[X]$ (e.g., $p_0 = X^2 + 1$). Hence p_0 splits in to linear factors $(\lambda - X)(\bar{\lambda} - X)$ in $\mathbb{C}[X]$, with $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

- (i) Show that if \mathbf{v} is a generalised eigenvector for λ with maximal height k , then $\bar{\mathbf{v}}$ is a generalised eigenvector for $\bar{\lambda}$ with maximal height k , and $[\mathbf{v}] \cap [\bar{\mathbf{v}}] = 0$. (Hint: Use Lemma 1.5.6.)
- (ii) Show that A is similar to a real matrix $K \in \mathbb{R}^{(n,n)}$ composed of just three kinds of (2×2) -blocks: $\mathbf{0} \in \mathbb{R}^{(2,2)}$, $E_2 \in \mathbb{R}^{(2,2)}$ and some $A_0 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathbb{R}^{(2,2)}$ with $b \neq 0$, where A_0 occurs along the diagonal, E_n and $\mathbf{0}$ immediately above the diagonal and just $\mathbf{0}$ everywhere else (a “block Jordan normal form”).

Hint: Put A into Jordan normal form over \mathbb{C} w.r.t. basis consisting of complex conjugate vector pairs; then combine such pairs to find a real basis as in exercise (E3.3).

- (iii) Give examples of $A_k \in \mathbb{R}^{(6,6)}$ with characteristic polynomial $(X^2 + 1)^3$ and minimal polynomials $q_{A_k} = (X^2 + 1)^k$ for $k = 1, 2, 3$.