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5th Exercise Sheet Linear Algebra II for MCS
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(E5.1) [Invariant subspaces and characteristic and minimal polynomial]

(i) Recall that $S_n = \mathrm{Sym}(\{1, 2, \ldots, n\})$ is the symmetric group on n elements. Prove that if $X \subseteq \{1, 2, \ldots, n\}$ is invariant under $\sigma \in S_n$ (so $\sigma(x) \in X$ for all $x \in X$), then so is $\overline{X} = \{1, 2, \ldots, n\} \backslash X$.

(ii) Let $M \in \mathbb{F}^{(n.n)}$ be a matrix of the form $\left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right)$ with $A \in \mathbb{F}^{(k,k)}, B \in \mathbb{F}^{(k,n-k)}$ and $C \in \mathbb{F}^{(n-k,n-k)}$. Show that $|M| = |A||C|$.

(iii) Let $\varphi \in \mathrm{Hom}(V, V)$ be an endomorphism of V and $U \subseteq V$ be a φ-invariant subspace. We write $\varphi_1 : U \to U$ for the restriction of φ to U. Show that

$$\varphi_2(\mathbf{v} + U) = \varphi(\mathbf{v}) + U$$

defines an endomorphism of V/U.

(iv) Show that if in the situation of (iii), V is finite dimensional, we have the following equality of the characteristic polynomials

$$p_\varphi = p_{\varphi_1} \cdot p_{\varphi_2}.$$

Does the same hold for the minimal polynomials (compare exercise E4.5)?

(E5.2) [Jordan Normal Form]

Write down matrices $A_i \in \mathbb{R}^{(4,4)}$ in Jordan normal form with the following properties:

(i) A_1 has eigenvalues 2 and 4, with 2 having algebraic multiplicity 3 and geometric multiplicity 1.

(ii) A_2 has the eigenvalue 5 with algebraic multiplicity 4 and geometric multiplicity 3.

(iii) A_3 has the eigenvalue 7 with algebraic multiplicity 2 and geometric multiplicity 2 and the eigenvalue -3 with algebraic multiplicity 2 and geometric multiplicity 1.

(iv) The matrices A_4 and A_5 both have the eigenvalue 1 with algebraic multiplicity 4 and geometric multiplicity 2 and have no other eigenvalues. Furthermore, A_4 and A_5 are not similar.

(E5.3) [Jordan normal form and minimal polynomial]

Let $A \in \mathbb{C}^{(5,5)}$ be a matrix with characteristic polynomial $p_A = (X-2)^2(X-3)^3$.

Find (up to permutation of Jordan blocks) all possible Jordan normal forms of A, and determine for all of these the corresponding minimal polynomial.

(E5.4) [Jordan normal form and characteristic polynomial]

Let $A \in \mathbb{C}^{(7,7)}$ be a matrix with minimal polynomial $q_A = X^3(X-1)^2$.

Find (up to permutation of Jordan blocks) all possible Jordan normal forms of A, and determine for all of these the corresponding characteristic polynomial.

(E5.5) [Jordan normal form and differential equations]

In exercise E1.4(v) we saw a method for solving the linear system of differential equations

$$\frac{d}{dt}\mathbf{x}(t) = A\mathbf{x}(t) \tag{1}$$

for a diagonalisable matrix $A \in \mathbb{C}^{(n.n)}$ (but the method works generally). We now follow a different method, which also works for any matrix $A \in \mathbb{C}^{(n,n)}$.

(i) Show that, if $\mathbf{v}$ is an eigenvalue of A with the eigenvalue λ, then $f : t \mapsto e^{\lambda t}\mathbf{v}$ is a solution of (1).

(ii) Show that, if λ is an eigenvalue and $\mathbf{v}$ is a solution of

$$(A - \lambda E)^n \mathbf{v} = \mathbf{0}$$

for some $n > 0$, then $f : t \mapsto e^{\lambda t}\left(\sum_{i=0}^{n-1} \frac{t^i}{i!}(A - \lambda E)^i \mathbf{v}\right)$ is a solution of (1).

(iii) Find a basis for the (three-dimensional) space of all solutions of (1) in case

$$A = \begin{pmatrix} -1 & -4 & 11 \\ 4 & 9 & -18 \\ 1 & 2 & -3 \end{pmatrix}.$$