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4th Exercise Sheet Linear Algebra II for MCS
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(E4.1) [Algebraic and geometric multiplicity]

Let φ be an endomorphism on a finite dimensional $\mathbb{F}$-vector space V, $\lambda \in \mathbb{F}$ an eigenvalue of φ. Show that the geometric multiplicity t of λ is smaller than or equal to its algebraic multiplicity s.

Hint: Find a basis B such that $[\![\varphi]\!]_B^B$ has the form $\left(\begin{array}{c|c} D & * \\ \hline 0 & * \end{array}\right)$, where $D = \lambda E_t$.

(E4.2) [Similar matrices]

Recall that $\mathrm{tr}(A)$ stands for the *trace* of an $n \times n$-matrix A, which is defined to be the sum over the diagonal entries: $\mathrm{tr}(A) = \sum_{1 \leqslant i \leqslant n} a_{ii}$ for $A = (a_{ij})_{1 \leqslant i,j \leqslant n} \in \mathbb{F}^{(n,n)}$.

(i) How does the characteristic polynomial p_A of a matrix A determine $\mathrm{tr}(A)$ and $|A|$? From this, conclude (again) that the trace is invariant under similarity (as is the determinant, of course).

(ii) Consider 2×2-matrices over the complex numbers. Why does their minimal polynomial determine their characteristic polynomial? Is the same true for 3×3-matrices?

Extra: Find two 2×2-matrices that are not similar, but have the same characteristic polynomial. Show that any two 2×2-matrices with the same minimal polynomial are similar in $\mathbb{C}^{(2,2)}$. Is the same true in $\mathbb{R}^{(2,2)}$? (Hint: for the last question, use Exercise E3.4.ii-iii.)

(iii) Discuss necessary and sufficient conditions (also in terms of the determinant, the trace and the minimal and characteristic polynomial of a matrix) for the similarity of two matrices. Use these criteria to split the following 9 matrices into equivalence classes w.r.t. similarity.

$$A_1 = \begin{pmatrix} 4 & 2 & 3 \\ 1 & 3 & 2 \\ 6 & 8 & 7 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 7 & 2 \\ 2 & 8 & 6 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$A_6 = \begin{pmatrix} 2 & 4 & 3 \\ 3 & 1 & 2 \\ 8 & 6 & 7 \end{pmatrix}, \quad A_7 = \begin{pmatrix} 4 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 2 & 2 \end{pmatrix}, \quad A_8 = \begin{pmatrix} 2 & 5 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{pmatrix}, \quad A_9 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(E4.3) [Cayley-Hamilton]

(i) Give an elementary proof (which means in particular: without using the Cayley-Hamilton Theorem) that every matrix is the root of a nontrivial polynomial, i.e. that there is a non-trivial polynomial $p(x)$ such that $p(A) = 0$ for the matrix A.

Hint: Consider $\mathbb{F}^{n,n}$ as a vector space over $\mathbb{F}$.

(ii) Prove the Cayley-Hamilton Theorem for 2×2 matrices by direct computation.

(E4.4) [Minimal Polynomials]

Consider the matrices
$$A := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 6 \end{pmatrix}, \qquad B := \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{pmatrix}.$$

(i) Determine the characteristic and minimal polynomials of A and B.

(ii) For matrix B:

(a) Show that $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ are eigenvectors with eigenvalue 2.

(b) Determine an eigenvector $\mathbf{v}_4$ with eigenvalue 3.

(c) Check that $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is a solution of $(B-2E_4)^2\mathbf{x} = \mathbf{0}$ and that $B\mathbf{v}_3 = 2\mathbf{v}_3 + \mathbf{v}_1$.

(d) Determine the matrix that represents φ_B w.r.t. the basis $B = (\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_4)$.

(E4.5) [Minimal Polynomials]

(i) Let A_1 and A_2 be quadratic matrices and let q_{A_1} and q_{A_2} be the corresponding minimal polynomials. Show that the minimal polynomial q_B of the matrix
$$B := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$
is the least common multiple of the polynomials q_{A_1} and q_{A_2}.

Remark: this observation generalises to arbitrary numbers of blocks by induction.

(ii) Determine the minimal polynomial q_B of
$$B := \begin{pmatrix} 2 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$