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May 3, 2007

3rd Exercise Sheet Linear Algebra II for MCS
Summer Term 2007

(E3.1) [Review and useful facts]

Recall that complex numbers are represented by expressions of the form

$$z = a + bi$$

with $a, b \in \mathbb{R}$, $i \notin \mathbb{R}$ a new constant. Identifying $a \in \mathbb{R}$ with the complex number $a+0i$ and the new constant i with $0 + 1i$, one may introduce addition and multiplication as the natural extensions of addition and multiplication in $\mathbb{R}$ based on associativity, commutativity, distributivity and the identity $i^2 = -1$. $\mathbb{R}$ thus becomes a subfield of the field of complex numbers.

(i) Let $z_1 = 3 + 4i$ and $z_2 = 5 + 12i$ be complex numbers. Compute

$$z_1^{-1}, \quad z_2^{-1}, \quad z_1^2, \quad z_2^2, \quad \text{and} \quad z_1 z_2,$$

and draw them in the complex plane. Find the complex square roots of i, z_1 and z_2, i.e., solve the equations $x^2 = i, x^2 = z_1, x^2 = z_2$ over $\mathbb{C}$.

(ii) Define for $\varphi \in \mathbb{R}$: $e^{i\varphi} := \cos\varphi + i \sin\varphi$.
Show that $e^{i\varphi}e^{i\psi} = e^{i(\varphi+\psi)}$ and $(e^{i\varphi})^n = e^{in\varphi}$ for every natural number n.

(iii) Show that every complex number $z \in \mathbb{C}\backslash\{0\}$ can be represented as:

$$z = re^{i\varphi},$$

with $r \in \mathbb{R}_{>0}$. Prove that this representation is unique in the sense that, when also $z = se^{i\psi}$ with $s > 0$, then $r = s$ and $\varphi \equiv \psi \mod 2\pi$.

(iv) Use the representation from (iii) to find all complex solutions of $z^5 = 1$ and draw these in the complex plane. In general, find all solutions to $z^n = w$ for $w \in \mathbb{C}\backslash\{0\}, n \in \mathbb{N}$.

(E3.2) [Polynomials over different fields]

Consider the following polynomials in $\mathbb{F}[X]$ for $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$:

$$p_1 = X^3 - 2, \qquad p_2 = X^3 + 4X^2 + 2X, \qquad p_3 = X^3 - X^2 - 2X + 2.$$

(i) Which of these polynomials are irreducible in $\mathbb{F}[X]$?

(ii) Which of these polynomials decompose into linear factors over $\mathbb{F}[X]$?

(iii) Suppose p_i is the characteristic polynomial of a matrix $A_i \in \mathbb{F}^{(3,3)}$. Which of the A_i is diagonalisable over $\mathbb{F}$?

(E3.3) [Complex reduction of real polynomials]

Recall that the function which sends a complex number $z = a+bi$ to its complex conjugate $\overline{z} = \overline{a+bi} = a - bi$, is a field automorphism that fixes $\mathbb{R}$ pointwise.

(i) (Compare Section 1.2.3 in the notes.) Use the decomposition of complex polynomials into linear factors over $\mathbb{C}[X]$ to decompose p into irreducible factors over $\mathbb{R}[X]$, where

$$p = X^5 - 5X^4 + 13X^3 - 25X^2 + 36X - 20.$$

Hint: p has complex roots 1, $2i$ and $2+i$, among others.

(ii) Let $A \in \mathbb{R}^{(2,2)} \subseteq \mathbb{C}^{(2,2)}$ be a real matrix, representing the linear map $\varphi \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ with respect to the standard basis. Assume that the associated characteristic polynomial has two distinct complex roots in $\mathbb{C}[X]$, but not in $\mathbb{R}[X]$. (This means that A is diagonalisable over $\mathbb{C}$, but not necessarily over $\mathbb{R}$.)

Let $\mathbf{v} \in \mathbb{C}^2$ be an eigenvector of A over $\mathbb{C}^2$ with eigenvalue λ. Show that $\overline{\mathbf{v}}$ (the component-wise complex conjugate) is an eigenvector with eigenvalue $\overline{\lambda}$. Show also that $\mathbf{b}_1 = \mathbf{v} + \overline{\mathbf{v}}$ and $\mathbf{b}_2 = i(\mathbf{v} - \overline{\mathbf{v}})$ are two linearly independent vectors in $\mathbb{R}^2$.

(iii) (Continuation of (ii).) Determine the real matrix C, similar to A, that represents φ with respect to the basis $(\mathbf{b}_1, \mathbf{b}_2)$ of $\mathbb{R}^2$. What is the geometric interpretation of the linear map in $\mathbb{R}^2$ that is represented, with respect to the standard basis, by C?

(E3.4) [Polynomials over $\mathbb{F}_2$]

(i) Show that in $\mathbb{F}_2[X]$ any non-linear polynomial with an odd number of powers X^i for $i \geq 1$ (with or without the constant term 1) is reducible.

(ii) Find in $\mathbb{F}_2[X]$ all irreducible polynomials of degree 3 and 4.

(E3.5) [Upper triangle shape]

Use the method in the proof of Proposition 1.3.1 to find a real upper triangular matrix similar to

$$A = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$