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2nd Exercise Sheet Linear Algebra II for MCS
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(E2.1) [Further properties of eigenvalues/eigenspaces]

Throughout this exercise, A is a square matrix with entries in $\mathbb{R}$.

(i) Prove or disprove that A and A^t have the same eigenvalues. (A^t is the transpose of A, see lecture notes for Linear Algebra I.)

(ii) Prove or disprove that A and A^t have the same eigenspaces.

(iii) Assume A is regular and let $\mathbf{v}$ be an eigenvector of A with eigenvalue λ. Show that $\mathbf{v}$ is also an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.

(iv) Let $\mathbf{v}$ be an eigenvector of the matrix A with eigenvalue λ and let s be a scalar. Show that $\mathbf{v}$ is an eigenvector of $A - sE$ with eigenvalue $\lambda - s$.

(v) For the following matrices, determine all eigenvalues and compute bases of the corresponding eigenspaces:

$$A = \begin{pmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{pmatrix}, \quad A^t, \quad A^{-1}, \quad A - 3E.$$

(E2.2) [Eigenvalues]

Let V be a finite dimensional vector space and φ, μ be endomorphisms of V.
Prove: if λ is an eigenvalue of $\varphi \circ \mu$, then it is an eigenvalue of $\mu \circ \varphi$ as well.

Hint: it may help to distinguish cases according to whether $\lambda \neq 0$ or $\lambda = 0$.

Extra: Can you give a counterexample in case V is infinite dimensional?

(E2.3) [Application of diagonalisation: Fibonacci Numbers, Golden Mean]

The sequence $f_0, f_1, f_2, \ldots$ of Fibonacci numbers is inductively defined as follows:

$$\begin{aligned} f_0 &= 0, \\ f_1 &= 1, \\ f_{k+2} &= f_{k+1} + f_k. \end{aligned}$$

(i) We define $\mathbf{u}_k = \binom{f_{k+1}}{f_k} \in \mathbb{R}^2$. Find a matrix A and a start vector $\mathbf{u}_0$ such that $\mathbf{u}_{k+1} = A\mathbf{u}_k$ for all k.

(ii) What are the eigenvalues of A? Give an explicit formula for f_k.

Hint: Use the eigenvalues λ_1 and λ_2 as abbreviations as long as possible.

(iii) Compute the limit $a = \lim_{k\to\infty} \frac{f_{k+1}}{f_k}$.

The limit is called the Golden Mean, it divides a line segment of length 1 into two parts a and $1-a$ such that $\frac{1}{a} = \frac{a}{1-a}$.

(E2.4) [Ideals]

Recall that a non-empty subset I of a commutative ring R is called an ideal, when it is closed under addition and under multiplication with arbitrary ring elements. The principal ideal I_a generated by a fixed element $a \in R$ is defined by

$$I_a = \{ra \,:\, r \in R\},$$

the set of all multiples of a (see Definition 1.2.16 on page 22 of the notes).

(i) Verify that I_a is the smallest ($\subseteq$-minimal) ideal containing a.

(ii) Let I and J be two ideals in a commutative ring R. Prove that

$$I + J = \{i + j : i \in I, j \in J\}$$

is again an ideal, in fact, the smallest ideal containing both I and J.

(iii) Prove that every ideal over $\mathbb{Z}$ is principal. Is the same true in the rings $\mathbb{Z}_n$ $(n \in \mathbb{Z})$?

(iv) For two elements $m, n \in \mathbb{Z}$, the set $I_m + I_n$ is an ideal over $\mathbb{Z}$, hence principal. This means that $I_m + I_n = I_k$ for some element $k \in \mathbb{Z}$. Express k in terms of m and n.

(v) For any two ideals I and J in a commutative ring R, find an expression for $I \wedge J$, the largest ideal contained in both I and J. Over the ring $\mathbb{Z}$, how does one determine for any pair $m, n \in \mathbb{Z}$ the $k \in \mathbb{Z}$ such that $I_m \wedge I_n = I_k$?

(E2.5) [Eigenvalues of nilpotent maps]

Let V be a vector space of dimension greater than 0, and let $\varphi : V \to V$ be a nilpotent endomorphism, that is, an endomorphism such that $\varphi^k = \mathbf{0}$ for some $k \in \mathbb{N}$.

(i) Show that 0 is the only possible eigenvalue of φ.

(ii) Show that 0 is an eigenvalue of φ.