

April 19, 2007

## 1st Exercise Sheet Linear Algebra II for MCS Summer Term 2007

### (E1.1) [Eigenvalues and eigenvectors]

In  $\mathbb{R}^3$ , let  $g$  be a line through the origin and  $E$  be a plane through the origin such that  $g$  is not in  $E$ . Determine (geometrically) the eigenvalues and eigenspaces of the following linear maps:

- (i) reflection in the plane  $E$ .
- (ii) reflection in the origin.
- (iii) parallel projection in the direction of  $g$  onto  $E$ .
- (iv) rotation about  $g$  through  $\frac{1}{3}\pi$  followed by rescaling in the direction of  $g$  with factor 6.

Which of these maps admit a basis of eigenvectors?

### (E1.2) [Geometric characterisation of linear maps by eigenvalues]

Give a geometric description of all the endomorphisms of  $\mathbb{R}^3$  with the following sets of eigenvalues:

- (i)  $\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 1$
- (ii)  $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$
- (iii)  $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2$

Note that you cannot assume anything about the corresponding eigenvectors other than that they form a basis (why?).

### (E1.3) [Eigenvalues and eigenvectors]

Consider the real  $2 \times 2$  matrix  $A = \begin{pmatrix} -2 & 6 \\ -2 & 5 \end{pmatrix}$ , and the linear map  $\varphi = \varphi_A$  given by  $A$  w.r.t. the standard basis.

- (i) Calculate the eigenvalues of  $A$  by expanding  $\det(A - \lambda E)$  and find the zeroes/roots of the characteristic polynomial.
- (ii) For each eigenvalue  $\lambda_i$  determine the eigenspace  $V_{\lambda_i}$ .
- (iii) Find a basis  $B$  of  $\mathbb{R}^2$  that only consists of eigenvectors of  $\varphi$  and find the matrix of the map  $\varphi$  with respect to the basis  $B$ .

**(E1.4) [Diagonalisation]**

Consider the matrix  $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$  over  $\mathbb{R}$ .

- (i) Determine all eigenvalues of  $A$  and corresponding eigenvectors.
- (ii) Find a regular matrix  $C$  such that  $D = C^{-1}AC$  is a diagonal matrix.
- (iii) Calculate  $A^6$ .
- (iv) Find a “positive square root” of  $A$ , i.e., find a matrix  $R$  with non-negative eigenvalues such that  $R^2 = A$
- (v) Check that  $t \mapsto e^{tA}\mathbf{v}_0$  solves the differential equation  $\frac{d}{dt}\mathbf{v}(t) = A\mathbf{v}(t)$  with initial value  $\mathbf{v}(0) = \mathbf{v}_0$ .

**(E1.5) [Dividing polynomials]**

In case you do not know this from school or from the analysis course: one can divide polynomials with remainders the very same way you do an ordinary written division of numbers, see also lecture notes, page 14.

Here is an example. We divide  $x^3 - 7x - 6$  by  $x - 1$ .

$$\begin{array}{r}
 x^3 + 0x^2 - 7x - 6 = (x - 1) \cdot (x^2 + x - 6) + 12 \\
 \underline{x^3 - x^2} \phantom{- 7x - 6} \\
 \phantom{x^3} x^2 - 7x \phantom{- 6} \\
 \phantom{x^3} \underline{x^2 - x} \phantom{- 6} \\
 \phantom{x^3} \phantom{x^2} - 6x - 6 \\
 \phantom{x^3} \phantom{x^2} \underline{- 6x + 6} \\
 \phantom{x^3} \phantom{x^2} \phantom{- 6x} 12
 \end{array}$$

- (i) Divide  $x^3 - 7x - 6$  by  $x + 1$ .
- (ii) Find all roots of  $x^3 - 7x - 6$ .