8. Paths

In the sequel, (X,d) is a metric space.

Definition 8.1

A (parametrized) path (arc, curve) in X is a continuous function $\gamma : I \to X$, where $I \subseteq \mathbb{R}$ is a nonempty interval.

The image $\gamma(I) = \{\gamma(t) : t \in I\} \subseteq X$ is also called a curve.

Remark 8.2

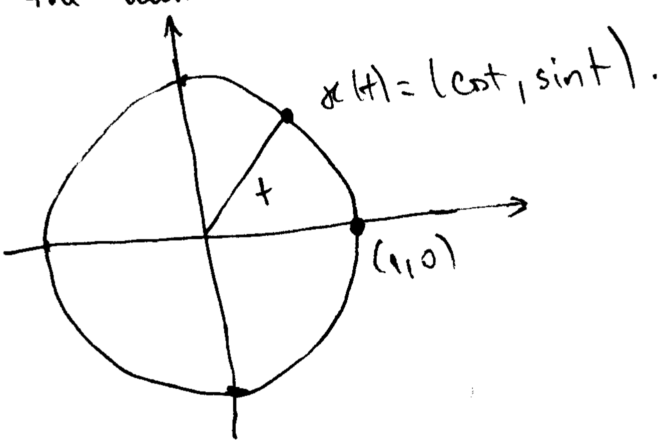
A path γ in $\mathbb{R}^n$ is just an n-tuple of continuous functions $\gamma_i : I \to \mathbb{R}$, $i = 1, \dots, n$ s.t. for all $t \in I$,

$$\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t)).$$

The functions γ_i, $i = 1, \dots, n$ are called the components of γ.

Example 8.3

(1) Consider the unit circle $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ in $\mathbb{R}^2$.



At time $t = 0$, a particle at the point $(1,0) \in C$ starts to move at constant speed along C in the anticlockwise direction (mathematically positive sense) and returns for the first time to this initial point at $t = 2\pi$. It is easy to see that at any time $t \in [0, 2\pi]$, the position of the particle may be given by $\varkappa(t) = (\cos t, \sin t)$. Thus, if we define the path $\varkappa$ by:

$$\varkappa : [0, 2\pi] \to \mathbb{R}^2, \quad \varkappa(t) = (\cos t, \sin t),$$

then $C = \varkappa([0, 2\pi])$, that is $\varkappa$ describes the unit circle.

(2) Consider the path $\tilde{\varkappa}$ in $\mathbb{R}^2$ defined by

$$\tilde{\varkappa} : [0, 2\pi] \to \mathbb{R}^2, \quad \tilde{\varkappa}(t) = (\cos t, -\sin t).$$

Again $\tilde{\varkappa}([0, 2\pi]) = C$, so $\tilde{\varkappa}$ describes the unit circle, but in the clockwise direction (mathematically negative sense).

Although $\varkappa([0, 2\pi]) = \tilde{\varkappa}([0, 2\pi]) = C$, the paths $\varkappa$ and $\tilde{\varkappa}$ are different.

(3) Let $f : I \to \mathbb{R}$ be a continuous function defined on an interval $I \subseteq \mathbb{R}$. Then the path

$$\varkappa : I \to \mathbb{R}^2, \quad \varkappa(t) = (t, f(t))$$

describes the graph of f, that is $\varkappa(I) = G_f$.

Remark 8.4

One sometimes writes $\varkappa(t) = (x(t), y(t))$ or $\varkappa(t) = (x(t), y(t), z(t))$ for paths in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

In the sequel we consider paths $\gamma : [a,b] \to X$ defined on a compact interval $[a,b]$, where $a < b \in \mathbb{R}$. If $\gamma(a) = x$ and $\gamma(b) = y$, we say that x and y are the <u>endpoints</u> of γ and that γ <u>joins</u> the points x and y.

Definition 8.5

Let $\gamma : [a,b] \to X$ be a path in X and $P = (a = t_0 < \ldots < t_n = b)$ be a partition of $[a,b]$. The <u>total variation of γ with respect to P</u> is defined as

$$V_P(\gamma) = \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})).$$

Definition 8.6

Let $\gamma : [a,b] \to X$ be a path in X. The <u>length</u> (<u>arc length</u>) of γ is given by

$$L(\gamma) = \begin{cases} \sup\{ V_P(\gamma) : P \text{ is a partition of } [a,b] \} & \text{if this supremum exists.} \\ \infty & \text{otherwise.} \end{cases}$$

Definition 8.7

A path $\gamma : [a,b] \to X$ is called <u>rectifiable</u> if its length is finite.

Proposition 8.8

Let $\gamma : [a,b] \to X$ be a path, $x = \gamma(a)$ and $y = \gamma(b)$.

(i) $d(x,y) \le L(\gamma)$

(ii) $L(\gamma) = 0 \iff \gamma$ is a constant path.

(iii) if γ is Lipschitz continuous with Lipschitz constant L, then

$$L(\gamma) \leq L(b-a).$$

<u>Proof</u>

(i) Consider the partition $P_0 = (a = t_0 < t_1 = b)$. Then $V_{P_0}(\gamma) = d(x,y)$, hence

$$L(\gamma) = \sup_P V_P(\gamma) \geq V_{P_0}(\gamma) = d(x,y).$$

(ii)

„$\Leftarrow$" If γ is a constant path, $\gamma(t) = c$ for all $t \in [a,b]$, then it is easy to see that $V_P(\gamma) = 0$ for any partition P of $[a,b]$.

Thus, $L(\gamma) = 0$.

„$\Rightarrow$" Assume that $L(\gamma) = 0$, that is $V_P(\gamma) = 0$ for any partition P of $[a,b]$. Let $t \in]a,b[$ and consider the partition $P = (a < t < b)$. Then $0 = V_P(\gamma) = d(\gamma(a), \gamma(t)) + d(\gamma(t), \gamma(b)) \geq 0$. It follows that we must have $\gamma(a) = \gamma(t) = \gamma(b)$, so γ is constant.

(iii) For any partition $P = (a = t_0 < t_1 < \dots < t_n = b)$ of $[a,b]$ we have that

$$V_P(\gamma) = \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \leq \sum_{i=0}^{n-1} L \cdot |t_i - t_{i+1}| =$$

$$= L \cdot \sum_{i=0}^{n-1} (t_{i+1} - t_i) = L(b-a).$$

Hence, $L(\gamma) \leq L(b-a)$.

<u>Definition 8.8</u>

Let $(V, \|\cdot\|)$ be a normed space. For any $x, y \in V$, the <u>**affine path joining x and y**</u> is the path $\gamma : [0,1] \to V$, $\gamma(t) = (1-t)x + ty$.

Note that an affine path is indeed a path, that is, it is continuous.

Proposition 8.10

Let $(V, \|\cdot\|)$ be a normed space and let $\gamma : [0,1] \to X$ be an affine path joining two points $x, y \in V$. Then

$$L(\gamma) = \|x-y\|.$$

Proof

For all t_1 and t_2 satisfying $0 \le t_1 \le t_2 \le 1$ we have

$$\|\gamma(t_1) - \gamma(t_2)\| = \|(1-t_1)x + t_1 y - (1-t_2)x - t_2 y\| = \|(t_2-t_1)x + (t_1-t_2)y\|$$
$$= \|(t_2-t_1)(x-y)\| = |t_2-t_1|\,\|x-y\| = (t_2-t_1)\|x-y\|.$$

Thus, if $P = (0 = t_0 < t_1 < \dots < t_n = 1)$ is an arbitrary partition of $[0,1]$ we get that

$$V_P(\gamma) = \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) = \sum_{i=0}^{n-1} \|\gamma(t_i) - \gamma(t_{i+1})\| =$$
$$= \sum_{i=0}^{n-1} (t_{i+1}-t_i)\cdot\|x-y\| = \|x-y\| \cdot \sum_{i=0}^{n-1} (t_{i+1}-t_i) = \|x-y\|(t_n - t_0)$$
$$= \|x-y\|.$$

$\square$

Definition 8.11 (Change of parameters)

Let $\gamma : [a,b] \to X$ and $\gamma' : [c,d] \to X$ be two paths in X. We say that γ' is **obtained from γ by a change of parameters** if there exists a function $\varphi : [c,d] \to [a,b]$ that is monotone, surjective and that satisfies $\gamma' = \gamma \circ \varphi$.

$$\begin{tikzcd}
{[c,d]} \arrow[r, "\varphi"] \arrow[dr, "\gamma'"'] & {[a,b]} \arrow[d, "\gamma"] \\
 & X
\end{tikzcd}$$

The map ψ is called the <u>change of parameters</u>.

Remark 8.12

(i) We do not require that the map ψ be injective.

(ii) A monotone and surjective map between two intervals is necessarily continuous. Thus, ψ is continuous.

Proposition 8.13 (<u>Length is invariant under change of parameters</u>)

Let $\gamma': [c,d] \to X$ be a path obtained from a path $\gamma: [a,b] \to X$ by a change of parameters. Then $L(\gamma) = L(\gamma')$.

Proof.

Exercise.

Definition 8.14

Two paths $\gamma: [a,b] \to X$ and $\gamma': [c,d] \to X$ are called <u>equivalent</u> if there exists a change of parameters $\psi: [c,d] \to [a,b]$ s.t. γ' is obtained from γ by ψ and, moreover, ψ is <u>strictly isotone</u>.

Remark 8.15

The binary relation which we have defined on the set of all paths in X in 8.14 is indeed an equivalence relation.

Proof

Exercise.

Corollary 8.16

Two equivalent paths have the same length and the same image.

Proof.

Let $\gamma : [a,b] \to X$, $\gamma' : [c,d] \to X$ be two equivalent paths and $\varphi : [c,d] \to [a,b]$ the strictly isotone change of parameter.

Then $L(\gamma) = L(\gamma')$, by Proposition 8.13. We get also that

$$\gamma'([c,d]) = (\gamma \circ \varphi)([c,d]) = \gamma(\varphi([c,d])) = \gamma([a,b]).$$

$\square$

Proposition 8.17 (**Additivity of lengths**)

Let $\gamma : [a,b] \to X$ be a path in X. For all $c \in [a,b]$,

$$L(\gamma) = L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]}).$$

Proof.

Exercise.

Definition 8.18 (**Concatenation of paths**)

Let $a, b, c \in \mathbb{R}$ be such that $a < c < b$. If $\gamma_1 : [a,c] \to X$ and $\gamma_2 : [c,b] \to X$ are two paths in X satisfying $\gamma_1(c) = \gamma_2(c)$, then we can define the path $\gamma_1 * \gamma_2 : [a,b] \to X$ by setting

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{if } a \le t \le c \\ \gamma_2(t) & \text{if } c \le t \le b. \end{cases}$$

The path $\gamma_1 * \gamma_2$ is called the **concatenation** of γ_1 and γ_2.

By the additivity of lengths, we get that $L(\gamma_1 * \gamma_2) = L(\gamma_1) + L(\gamma_2)$.

Differentiable paths in $\mathbb{R}^n$

In the sequel, we consider the Euclidean space of dimension n, that is $(\mathbb{R}^n, \|\cdot\|_2)$, where $\|\cdot\|_2$ is the Euclidean norm:

$$\|(x_1,\dots,x_n)\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}\ .$$

Definition 8.19

A path $\gamma: I \to \mathbb{R}^n$, $t \mapsto \gamma(t) = (\gamma_1(t),\dots,\gamma_n(t))$ is called **differentiable** iff all components $\gamma_1,\dots,\gamma_n$ are differentiable.

The path γ is called **continuously differentiable** iff all components are continuously differentiable.

Definition 8.20

Let $\gamma: I \to \mathbb{R}^n$, $\gamma = (\gamma_1,\dots,\gamma_n)$ be a differentiable path in $\mathbb{R}^n$.

For $t \in I$,

$$\gamma'(t) = (\gamma_1'(t),\dots,\gamma_n'(t))$$

is called the **tangential vector** of the path γ at the parameter value t.

If $\gamma'(t) \neq 0$, then γ is called **regular** at the parameter value t. In this case, the normalized vector $\dfrac{\gamma'(t)}{\|\gamma'(t)\|_2}$ is called the **tangential unit vector** at the parameter value t.

The path γ is called **regular** iff $\gamma'(t) \neq 0$ for all $t \in I$.

Theorem 8.21

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a continuously differentiable path in $\mathbb{R}^n$.

Then γ is rectifiable and

$$L(\gamma) = \int_a^b \|\gamma'(t)\|_2 \, dt .$$

Proof

On page 10.

Corollary 8.22

Let $f : [a,b] \to \mathbb{R}$ be a continuously differentiable function and let the path $\gamma : [a,b] \to \mathbb{R}^2$ be defined as the graph of f:

$$\gamma(t) = (t, f(t)) \qquad \text{for all } t \in [a,b].$$

Then γ is rectifiable and

$$L(\gamma) = \int_a^b \sqrt{1 + f'(t)^2} \, dt .$$

Proof

γ is continuously differentiable, $\gamma'(t) = (1, f'(t))$ for all $t \in [a,b]$, so $\|\gamma'(t)\|_2 = \sqrt{1 + f'(t)^2}$. Apply Theorem 8.21

$\square$

Example 8.23

Consider the curve $\gamma : [0,2\pi] \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \sin t)$ from Example 8.3(1). Then for all $t \in [0,2\pi]$,

$$\gamma'(t) = (-\sin t, \cos t) \text{ and } \|\gamma'(t)\|_2 = \sqrt{(-\sin t)^2 + \cos^2 t} = 1.$$

Thus, γ is rectifiable and

$$L(\gamma) = \int_0^{2\pi} 1 \, dt = 2\pi.$$

Proof of Theorem 8.21

Let $\gamma = (\gamma_1, \dots, \gamma_n)$. Then $\|\gamma'(t)\|_2 = \sqrt{\sum_{i=1}^{n} \gamma_i'(t)^2}$ for all $t \in [a,b]$.

For all $t \in [a,b]$, we set $\gamma_t = \gamma|_{[a,t]}$ and we consider the map

$$s : [a,b] \to \mathbb{R}, \quad s(t) = L(\gamma_t).$$

Then $s(a) = 0$ and $s(b) = L(\gamma)$. We will show that s is differentiable and that its derivative is equal to $\|\gamma'(t)\|_2$. As a consequence, by using the Fundamental theorem of Integral and Differential Calculus, we get that

$$L(\gamma) = s(b) - s(a) = \int_a^b s'(t)\,dt = \int_a^b \|\gamma'(t)\|_2 \, dt.$$

Firstly, let us remark that as a consequence of Proposition 8.17, s is increasing and for all t, t' satisfying $a \le t < t' \le b$, we have

(1) $$s(t') - s(t) = L(\gamma|_{[t,t']}).$$

Let us fix a real number $\varepsilon > 0$.

Claim

There exists $\delta > 0$ s.t. for all t and t' satisfying $a \le t < t' \le b$ and $t - t' < \delta$ we have, for all $j = 1, \dots, n$ and for all $\tau \in [t, t']$

$$x_j'(\tau)^2 \le x_j'(t)^2 + \varepsilon.$$

Proof of the claim

For any $j = 1, \dots, n$, the function $x_j' : [a, b] \to \mathbb{R}$ is continuous. Since $[a, b]$ is compact, it follows that x_j' is bounded, so there exists $M_j > 0$ s.t. $|x_j'(t)| \le M_j$ for all $t \in [a, b]$.

Let $M := \max \{ M_j : j = 1, \dots, n \}$.

Furthermore, x_j' is uniformly continuous, so there exists $\delta_j > 0$ s.t.

$$(2) \quad \forall t', t \in [a, b] \left(|t - t'| < \delta_j \Rightarrow |x_j'(t) - x_j'(t')| < \frac{\varepsilon}{2M} \right)$$

Take $\delta := \min\limits_{j = 1, \dots, n} \delta_j$.

Let now t, t' be s.t. $a \le t < t' \le b$ and $t - t' < \delta$, let $\tau \in [t, t']$ and $j = 1, \dots, n$ be arbitrary.

Since $t \le \tau \le t'$, we have that $0 \le \tau - t \le t' - t < \delta$, so that $|\tau - t| < \delta \le \delta_j$. By (2), it follows that $|x_j'(\tau) - x_j'(t)| < \frac{\varepsilon}{2M}$.

Hence,

$$x_j'(\tau)^2 - x_j'(t)^2 \le |x_j'(\tau)^2 - x_j'(t)^2| = |(x_j'(\tau) - x_j'(t))(x_j'(\tau) + x_j'(t))|$$
$$\le |x_j'(\tau) - x_j'(t)| \left(|x_j'(\tau)| + |x_j'(t)| \right) <$$
$$< \frac{\varepsilon}{2M} \cdot 2M = \varepsilon. \qquad \square$$

Now let us take t and t' satisfying $a \le t < t' \le b$ and $t'-t < \delta$.

Let $P = (t = t_0 < \dots < t_k = t')$ be an arbitrary partition of $[t,t']$.

We have

$$V_P(\gamma|_{[t,t']}) = \sum_{i=0}^{k-1} \|\gamma(t_i) - \gamma(t_{i+1})\|_2 = \sum_{i=0}^{k-1} \sqrt{\sum_{j=1}^{n} (\gamma_j(t_i) - \gamma_j(t_{i+1}))^2}$$

By the mean value theorem, for all $i = 0, \dots, k-1$ and for all $j = 1, \dots$

we can find $\tau_{i,j} \in [t_i, t_{i+1}]$ s.t.

$$\gamma_j(t_i) - \gamma_j(t_{i+1}) = \gamma_j'(\tau_{i,j})(t_i - t_{i+1}).$$

It follows that

$$\begin{aligned}\sum_{j=1}^{n} (\gamma_j(t_i) - \gamma_j(t_{i+1}))^2 &= \sum_{j=1}^{n} \gamma_j'(\tau_{i,j})^2 (t_{i+1} - t_i)^2 \\ &\le \sum_{j=1}^{n} (\gamma_j'(t)^2 + \varepsilon)(t_{i+1} - t_i)^2 = \left(n\varepsilon + \sum_{j=1}^{n} \gamma_j'(t)^2\right)\cdot(t_{i+1} - t_i)^2 \\ &= \left(n\varepsilon + \|\gamma'(t)\|_2^2\right)(t_{i+1} - t_i)^2.\end{aligned}$$

Thus, we obtain

$$V_P(\gamma|_{[t,t']}) \le \sum_{i=0}^{k-1} \sqrt{n\varepsilon + \|\gamma'(t)\|_2^2}\cdot(t_{i+1} - t_i) = \sqrt{n\varepsilon + \|\gamma'(t)\|_2^2}\cdot(t'-t)$$

The right hand side in the last expression does not depend on the choice of the partition P. Thus, we have that

$$L(\gamma|_{[t,t']}) \le \sqrt{n\varepsilon + \|\gamma'(t)\|_2^2}\cdot(t'-t).$$

Using (1), we therefore obtain

$$(3) \qquad \frac{s(t') - s(t)}{t'-t} = \frac{L(\gamma|_{[t,t']})}{t'-t} \le \sqrt{n\varepsilon + \|\gamma'(t)\|_2^2}.$$

On the other hand, we have, using Prop. 8.8 (i),

$$\|\gamma(t') - \gamma(t)\|_2 \le L(\gamma|_{[t,t']}) = s(t') - s(t).$$

Hence,

$$(4) \quad \left\| \frac{x(t') - x(t)}{t'-t} \right\|_2 = \frac{\|x(t') - x(t)\|_2}{t'-t} \leq \frac{s(t') - s(t)}{t'-t}.$$

We have obtained that for all $\varepsilon > 0$,

$$(5) \quad \left\| \frac{x(t') - x(t)}{t'-t} \right\|_2 \leq \frac{s(t') - s(t)}{t'-t} \leq \sqrt{n\varepsilon + \|x'(t)\|_2^2}$$

holds for all $t, t' \in [a, z]$ s.t. $t \neq t'$, $|t - t'| < \delta$.

Letting ε tend to 0 in (5), we get

$$\left\| \frac{x(t') - x(t)}{t'-t} \right\|_2 \leq \frac{s(t') - s(t)}{t'-t} \leq \|x'(t)\|_2 .$$

Since
$$\lim_{t' \to t} \left\| \frac{x(t') - x(t)}{t'-t} \right\|_2 = \left\| \lim_{t' \to t} \frac{x(t') - x(t)}{t'-t} \right\|_2$$
$$= \left\| \left(\lim_{t' \to t} \frac{x_1(t') - x_1(t)}{t'-t}, \dots, \lim_{t' \to t} \frac{x_n(t') - x_n(t)}{t'-t} \right) \right\|_2$$
$$= \|(x_1'(t), \dots, x_n'(t))\|_2 = \|x'(t)\|_2,$$

it follows that
$$s'(t) = \lim_{t' \to t} \frac{s(t') - s(t)}{t'-t} = \|x'(t)\|_2$$

□