7. Sequences and series of functions

Let X be a nonempty set, $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n: X \to \mathbb{K}$ and $f: X \to \mathbb{K}$.

Definition 7.1

We say that (f_n) converges pointwise to f if for each $x \in X$, the sequence $(f_n(x))$ converges to $f(x)$ in $\mathbb{K}$.

We call f the pointwise limit of (f_n).

Remark 7.2

(f_n) converges pointwise to f

$$\Updownarrow$$

$$(\forall \varepsilon > 0)\ (\forall x \in X)\ (\exists N \in \mathbb{N})\ (\forall n \geq N)\ \left(|f_n(x) - f(x)| < \varepsilon \right).$$

Definition 7.3

We say that (f_n) converges uniformly to f if

$$(\forall \varepsilon > 0)\ (\exists N \in \mathbb{N})\ (\forall n \geq N)\ (\forall x \in X)\ \left(|f_n(x) - f(x)| < \varepsilon \right).$$

Note that for pointwise convergence the number $N \in \mathbb{N}$ may depend on $x \in X$, but for uniform convergence it must be possible to choose N independently of $x \in X$. As a consequence, we get:

Proposition 7.4

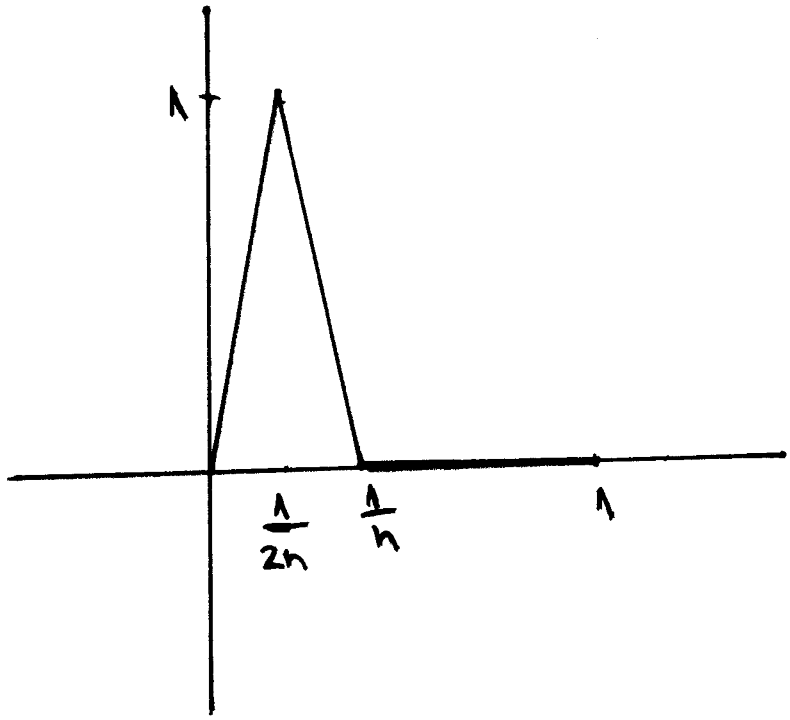
If (f_n) converges uniformly to f, then (f_n) converges pointwise to f.

The following example shows that the converse is not true.

Example 7.5

For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be the function defined by

$$f_n(x) = \begin{cases} 2nx & \text{for } x \in [0, \frac{1}{2n}[\\ 2-2nx & \text{for } x \in [\frac{1}{2n}, \frac{1}{n}[\\ 0 & \text{for } x \in [\frac{1}{n}, 1[. \end{cases}$$



Then
(i) (f_n) converges pointwise to the constant 0-function.
(ii) (f_n) does not converge uniformly.

Proof
(i) We have to prove that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$.

Case 1: $x = 0$. Then $f_n(x) = 0$ for all $n \in \mathbb{N}$, so $f_n(x) \to 0$.

Case 2: $x \in]0,1]$. Then there is $N \in \mathbb{N}$ s.t. $N \geq \frac{1}{x}$, that is $x \in [\frac{1}{N}, 1]$.

It follows that for all $n \geq N$ we have that $x \in [\frac{1}{n}, x]$, so $f_n(x) = 0$.

Hence, $\lim_{n\to\infty} f_n(x) = 0$.

(ii) Suppose that (f_n) converges uniformly. Then, by Prop. 7.4, (f_n) must converge uniformly to 0. Let $\varepsilon = \frac{1}{2}$. Then there exists $N \in \mathbb{N}$ s.t. $(\forall n \geq N)(\forall x \in [0,1])\left(|f_n(x)| < \varepsilon\right)$.

Consider $n := N$ and $x := \frac{1}{2N}$. It follows that

$f_n(x) = 1 > \varepsilon$, which is a contradiction. $\square$

Remark 7.6

If (f_n) is a sequence in $B(X)$ and $f \in B(X)$, then

$$(f_n) \text{ converges uniformly to } f \iff \lim_{n\to\infty} \|f_n - f\|_\infty = 0$$
$$\iff \lim_{n\to\infty} f_n = f \text{ in } (B(X), \|\cdot\|_\infty).$$

Theorem 7.7 (Uniform convergence and continuity)

Let (X,d) be a metric space, $f, f_n : X \to \mathbb{R}$ $(n \in \mathbb{N})$. Assume that (f_n) converges uniformly to f. Then

$$(\forall n \in \mathbb{N})(f_n \text{ is continuous}) \implies f \text{ is continuous}$$

Proof

See the proof of Theorem 6.16 (ii).

Theorem 7.8 (Uniform convergence and integrability)

Let (f_n) be a sequence of integrable functions $f_n : [a,b] \to \mathbb{R}$. Assume that (f_n) converges uniformly to a function $f : [a,b] \to \mathbb{R}$. Then f is integrable as well and

$$\int_a^b f(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

Proof

Since (f_n) converges uniformly to f, there exists $N \in \mathbb{N}$ s.t.

$$(\forall n \geq N)\,(\forall x \in X)\left(|f_n(x) - f(x)| < \frac{\varepsilon}{4(b-a)}\right),$$

that is

$$(\forall n \geq N)\,(\forall x \in X)\left(f_n(x) - \frac{\varepsilon}{4(b-a)} < f(x) < f_n(x) + \frac{\varepsilon}{4(b-a)}\right).$$

The integrability of f_N yields, by the Riemann Criterion, the existence of step functions $s, t \in S[a,b)$ s.t. $s \leq f_N \leq t$ and $\int_a^b (t-s) < \frac{\varepsilon}{2}$.

Then

$$s - \frac{\varepsilon}{4(b-a)} \leq f_N - \frac{\varepsilon}{4(b-a)} < f < f_N + \frac{\varepsilon}{4(b-a)} \leq t + \frac{\varepsilon}{4(b-a)}$$

and

$$\int_a^b \left(t + \frac{\varepsilon}{4(b-a)}\right) - \left(s - \frac{\varepsilon}{4(b-a)}\right) = \int_a^b (t-s) + \int_a^b \frac{\varepsilon}{2(b-a)} = \int_a^b (t-s) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, there exist step functions $s', t' \in S[a,b]$, $s' := s - \frac{\varepsilon}{4(b-a)}$, $t' := t + \frac{\varepsilon}{4(b-a)}$ s.t. $s' < f < t'$ and $\int (t'-s') < \varepsilon$. Applying the Riemann Criterion we get that f is also integrable.

Finally, for all $n \in \mathbb{N}$ we have that

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| \leq \int_a^b |f_n(x) - f(x)|\,dx \leq \int_a^b \|f_n - f\|_\infty\,dx$$
$$= (b-a)\|f_n - f\|_\infty.$$

Since $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$, we get that $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$. □

Definition 7.9

Let X be a nonempty set, (f_n) be a sequence of functions $f: X \to \mathbb{K}$ and $s_n = \sum_{k=1}^{n} f_k$ for all $n \in \mathbb{N}$.

The series of functions $\sum_{n=1}^{\infty} f_n$ is said to be pointwise convergent if the sequence $(s_n)_{n\in\mathbb{N}}$ is pointwise convergent. In this case, the sum of the series is the function $s: X \to \mathbb{K}$, $s(x) = \lim_{n\to\infty} s_n(x)$.

The series of functions $\sum_{n=1}^{\infty} f_n$ is called uniformly convergent if (s_n) is uniformly convergent.

Theorem 7.10 (Weierstraß Criterion)

Let $\sum_{n=1}^{\infty} f_n$ be a series of bounded functions $f_n: X \to \mathbb{K}$ and let $\sum_{n=1}^{\infty} a_n$ be a convergent series of nonnegative real numbers. If there exists $N \in \mathbb{N}$ s.t. $\|f_n\|_\infty \le a_n$ for all $n \geq N$, then the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent.

Proof

Let $s_n = \sum_{k=1}^{n} f_k$. We have to prove that (s_n) is uniformly convergent. Since (s_n) is a sequence in $B(X)$, this is equivalent with (s_n) is convergent in $(B(X), \|\cdot\|_\infty)$. By Theorem 6.16(i), $(B(X), \|\cdot\|_\infty)$ is a Banach space. Hence, it suffices to prove that (s_n) is Cauchy.

Let $\varepsilon > 0$. Applying the Cauchy Criterion (Theorem 2.48) for the convergent series $\sum_{n=1}^{\infty} a_n$, we get the existence of $K \in \mathbb{N}$ s.t.

$(\forall n > K)(\forall p \in \mathbb{N})\left(a_{n+1} + \dots + a_{n+p} < \varepsilon \right).$

Let now $J := \max(K, N)$. It follows that for all $n \geq J$ and all $p \in \mathbb{N}$,

$$\|s_{n+p} - s_n\|_\infty = \|f_{n+1} + \dots + f_{n+p}\|_\infty \leq \sum_{k=1}^{p} \|f_{n+k}\|_\infty \leq$$

$$\leq \sum_{k=1}^{p} a_{n+k} < \varepsilon.$$

Hence, (s_n) is a Cauchy sequence in $(B(X), \|\cdot\|_\infty)$

□

Example 7.11

The series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is uniformly convergent on $\mathbb{R}$, since for all $x \in \mathbb{R}$, $n \in \mathbb{N}$,

$$\left| \frac{\sin nx}{n^2} \right| \leq \frac{1}{n^2}$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Theorem 7.12

Let (f_n) be a sequence of continuous functions $f_n : [a,b] \to \mathbb{R}$. Assume that the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent with sum s. Then s is continuous and

$$\int_a^b s(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx.$$

Proof

Use Theorems 7.7 and 7.8