Anisotropic mesh adaptation based upon a posteriori error estimates

WEIZHANG HUANG, LENNARD KAMENSKI, AND JENS LANG

Abstract. An anisotropic mesh adaptation strategy for finite element solution of elliptic differential equations is considered. The adaptation method generates anisotropic adaptive meshes as quasi-uniform ones in some metric space. The associated metric tensor is computed by means of a posteriori hierarchical error estimates. A global hierarchical error estimate is employed in this study to obtain reliable directional information of the solution. The exact solution of the corresponding global error problem can be very costly. However, numerical results show that a few iterations of the symmetric Gauss-Seidel method are sufficient for obtaining a reasonably good approximation to the error for use in anisotropic mesh adaptation. The new method is compared with several strategies using local error estimators or recovered Hessians. Numerical results are presented for a selection of test examples and a mathematical model for heat conduction in a thermal battery with large orthotropic jumps in the material coefficients.

1. Introduction

Anisotropic mesh adaptation has proved to be a useful tool in numerical solution of partial differential equations (PDEs). This is especially true when problems arising from science and engineering have distinct anisotropic features. The ability to adapt the size, shape, and orientation of mesh elements according to certain quantities of interest can significantly improve the accuracy of the solution and enhance the computational efficiency.

Criteria for an optimal anisotropic triangular mesh were already given by D’Azevedo [6] and Simpson [27] in the early nineties of the last century. A number of algorithms for automatic construction of such meshes have since been developed.

A common approach for generating an anisotropic mesh is based on generation of a quasi-uniform mesh in some metric space. A key component of the approach is the determination of an appropriate metric often based on some type of error estimates. Unfortunately, classic error estimates do not suit this purpose well because they generally do not take the directional effect of the error or solution derivatives into account.

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consideration. This explains the recent interest in anisotropic error estimation; for example, see anisotropic interpolation error estimates by Formaggia and Perotto [12] and Huang [16, 19]. Such error estimates for numerical solution of PDEs can be found, among others, in works by Apel [2], Kunert [20], Formaggia and Perotto [13], and Picasso [26].

It is worth pointing out that most existing anisotropic error estimates are a priori, requiring information of the exact solution of either the underlying problem or its adjoint, which is typically unavailable in a numerical simulation. A widely-used approach of avoiding this difficulty in practical computation is to replace the information by one recovered from the obtained numerical approximation. A number of recovery techniques can be used for this purpose, such as the gradient recovery technique by Zienkiewicz and Zhu [31, 32] and the technique based on the variational formulation by Dolejší [9]. Zhang and Naga [30] have recently proposed a new algorithm to reconstruct the gradient (which can also be used to reconstruct the Hessian) by fitting a quadratic polynomial to the nodal function values and subsequently differentiating it. It has been shown by Zhang and Naga [30] and by Vallet et al. [28] that the latter is robust and works best among several recovery techniques. Generally speaking, recovery methods work well when exact nodal function values are used but may lose some accuracy when applied to finite element approximations on non-uniform meshes. Nevertheless, the optimality of mesh adaptation based on those recovered approximations can still be proven under suitable conditions, see Vassilevski and Lipnikov [29]. More recently, conditions for asymptotically exact gradient and convergent Hessian recovery from a hierarchical basis error estimator have been given by Ovall [24]. His result is based on superconvergence results by Bank and Xu [4, 5], which require that the mesh be uniform or almost uniform.

The objective of this paper is to study the use of a posteriori error estimates in anisotropic mesh adaptation. Although a posteriori error estimates are frequently used for mesh adaptation, especially for refinement strategies and recently also for construction of equidistributing meshes for numerical solution of two-point boundary value problems by He and Huang [14] as well as in connection with the moving finite element method by Lang et al. [22], up to now they have been used mostly for isotropic mesh adaptation. Moreover, Dobrowolski, Gräf and Pflaum [8] have pointed out that local error estimation can be inaccurate on anisotropic meshes. This shortcoming of local error estimates can be explained by the fact that they generally do not contain enough directional information of the solution, which is global in nature, and that their accuracy and effectiveness are sensitive to the aspect ratio of elements, which can be large for anisotropic meshes. We thus choose to develop our approach based on global error estimation. To enhance the computational efficiency, we employ an iterative
method to obtain a cost-efficient approximation to the solution of the corresponding globally defined error problem. Numerical results show that a few symmetric Gauss-Seidel iterations are sufficient for this purpose. This is not surprising since the approximation is used only in mesh generation and it is often unnecessary to compute the mesh to a very high accuracy as for the solution of the underlying differential equation. Numerical experiments also show that the new approach is comparable in accuracy and efficiency to methods using Hessian recovery. We also test it with a more challenging example: a heat conduction problem for a thermal battery with large and orthotropic jumps in the material coefficients.

The outline of the paper is as follows. In Section 2 the framework of using a posteriori hierarchical error estimates for anisotropic mesh adaptation in finite element approximation is described. In Section 3 the optimal metric tensor based on the interpolation error is developed. Several implementation issues are addressed in Section 4. Numerical results obtained with the new approach and with Hessian recovery based methods are presented in Section 5 for a selection of test examples. Numerical results for the heat conduction problem are given in Section 6. Finally, Section 7 contains conclusions and comments.

2. MODEL PROBLEM AND ADAPTIVE FINITE ELEMENT APPROXIMATION

2.1. Model problem and finite element approximation. Consider the variational problem

\[(P) \quad \begin{cases} \text{Find } u \in V \text{ such that} \\ a(u, v) = F(v), \quad \forall v \in V \end{cases}\]

where \( V \) is an appropriate Hilbert space of functions over a domain \( \Omega \subset \mathbb{R}^2 \), \( a(\cdot, \cdot) \) is a bilinear form defined on \( V \times V \), and \( F(\cdot) \) is a linear functional on \( V \). The finite element approximation \( u_h \) of \( u \) is the solution of the corresponding variational problem on a finite dimensional subspace \( V_h \subset V \), i.e.,

\[(P_h) \quad \begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = F(v_h), \quad \forall v_h \in V_h. \end{cases}\]

If the bilinear form \( a(\cdot, \cdot) \) is coercive and continuous on \( V \), both variational problems \( (P) \) and \( (P_h) \) have unique solutions. The finite dimensional subspace \( V_h \) is often chosen as a space of piecewise polynomials associated with a given mesh, say \( T_h \), on \( \Omega \). The variational problem \( (P_h) \) results in a system of \( \text{dim}(V_h) \) linear algebraic equations.

\(^1\)A Sandia National Laboratories benchmark problem.
space with a metric tensor $M$. The process is repeated until a good adaptation is achieved. An exam-

is to recover an approximate Hessian from the computed solution. We solve the underlying problem \[12, 17\], which is often unavailable in practical computation. The common approach to avoid this difficulty is to recover an approximate Hessian from the computed solution. We consider here an alternative approach, which uses an a posteriori error estimator for defining and computing $M_h$.

2.2. Adaptive linear finite element solution. In this work we consider a linear finite element method, where $V$ is taken as $H^1(\Omega)$ and $V_h$ is the space of continuous, piecewise linear functions over $T_h$. The approach proposed in this work can be applied to other finite elements without major modifications.

Let $T_h^{(i)} (i = 0, 1, ...)$ be an affine family of simplicial meshes on $\Omega$ and $V_h^{(i)}$ the corresponding space of continuous, piecewise linear functions. The adaptive solution is the result of an iterative process described as follows.

We start with an initial mesh $T_h^{(0)}$. On every mesh $T_h^{(i)}$ we solve the variational problem $(P_h)$ with $V_h^{(i)}$ and use the obtained approximation $u_h^{(i)}$ to compute a new adaptive mesh for the next iteration step. The new mesh $T_h^{(i+1)}$ is generated as an almost uniform mesh in a metric space with a metric tensor $M_h^{(i)}$ defined in terms of $u_h^{(i)}$. This yields the sequence

\[
(T_h^{(0)}, V_h^{(0)}) \rightarrow u_h^{(0)} \rightarrow M_h^{(0)} \rightarrow (T_h^{(1)}, V_h^{(1)}) \rightarrow u_h^{(1)} \rightarrow M_h^{(1)} \rightarrow \ldots
\]

The process is repeated until a good adaptation is achieved. An example of such adaptive meshes is shown in Fig. 1.
2.3. Mesh adaptation based on a posteriori error estimates. Let $R_h$ be a reconstruction operator applied to the numerical approximation $u_h$. It can be either a recovery process, a smoothing operator, or an operator connected to an a posteriori error estimate. We assume that the reconstruction $R_h u_h$ is better than $u_h$ in the sense that

$$\| R_h u_h - u \| \leq \beta \| u_h - u \|,$$

for a given norm $\| \cdot \|$, where $0 \leq \beta < 1$ is a constant. When $u_h$ is a linear finite element approximation and $R_h u_h$ is a quadratic finite element approximation, inequality (1) is known as the saturation assumption. See Section 2.5 for a more detailed discussion.

From the triangle inequality we immediately have

$$\| u - u_h \| \leq \frac{1}{1 - \beta} \| R_h u_h - u_h \|.$$

If the reconstruction $R_h$ has the property

$$I_h R_h v_h = v_h \quad \forall v_h \in V_h$$

for some interpolation operator $I_h$, we can bound the finite element approximation error by the (explicitly computable) interpolation error of the reconstructed function $R_h u_h$, viz.,

$$\| u - u_h \| \leq \frac{1}{1 - \beta} \| R_h u_h - u_h \| = \frac{1}{1 - \beta} \| R_h u_h - I_h R_h u_h \|.$$

Moreover, from the interpolation theory we know that the interpolation error for a given function $v$ can be bounded by a term depending on the triangulation $T_h$ and derivatives of $v$, i.e.,

$$\| v - I_h v \| \leq C \cdot IE(T_h, v),$$

where $C$ is a constant independent of $T_h$ and $v$. Therefore, we can rewrite (3) as

$$\| u - u_h \| \leq \frac{C}{1 - \beta} IE(T_h, R_h u_h).$$

In other words, up to a constant, the solution error is bounded by the interpolation error of $R_h u_h$.

2.4. Hierarchical basis. One possibility to achieve the property (2) is to use the hierarchical decomposition of the finite element space. Let

$$\bar{V}_h = V_h \oplus W_h,$$

where $W_h$ is a hierarchical extension of $V_h$ to $\bar{V}_h$. Each $\bar{v}_h \in \bar{V}_h$ has a unique representation $\bar{v}_h = v_h + w_h$ with $v_h \in V_h$ and $w_h \in W_h$. If an interpolation operator, $I_h : \bar{V}_h \mapsto V_h$, can be defined such that

$$I_h w_h = 0, \quad \forall w_h \in W_h$$

and if we define $R_h$ as

$$R_h u_h := u_h + z_h$$
for some $z_h \in W_h$, then we shall have the property (2) and the estimate (3). Moreover,

$$\| R_h u_h - \bar{I}_h R_h u_h \| = \| u_h + z_h - u_h \| = \| z_h \| = \| z_h - \bar{I}_h z_h \|.$$ 

Consequently, we can estimate the finite element approximation error by evaluating the interpolation error of $z_h$, i.e.,

$$\| u - u_h \| \leq \frac{1}{1 - \beta} \| z_h - \bar{I}_h z_h \| \leq \frac{C}{1 - \beta} IE(T_h, z_h).$$ (8)

In the context of a posteriori error estimates, $z_h$ is typically taken as a hierarchical basis error estimator.

2.5. A posteriori error estimate based on hierarchical basis.
The computation of the error estimator is based on a general framework, details on which can be found among others in the work of Bank and Smith [3] or Deuflhard, Leinen and Yserentant [7]. The approach is briefly explained as follows.

Let $u_h \in V_h$ be a linear finite element solution of the variational problem $(P_h)$ and let $\bar{V}_h = V_h \oplus W_h$, where $W_h$ is the linear span of the edge bubble functions. Obviously, $\bar{V}_h$ is a subspace of piecewise quadratic functions. Moreover, we can define $\bar{I}_h$ as the vertex-based, piecewise linear Lagrange interpolation. This interpolation satisfies (6) since the edge bubble functions vanish at vertices.

Let $e_h = u - u_h$ be the error of the finite element solution $u_h$. Then for all $v \in V$ we have

$$a(e_h, v) = F(v) - a(u_h, v).$$ (9)

The error estimate $z_h$ is then defined as the solution of the approximate error problem

$$\begin{cases} 
\text{Find } z_h \in W_h \text{ such that} \\
\ a(z_h, w_h) = F(w_h) - a(u_h, w_h) \quad \forall w_h \in W_h.
\end{cases} \ (E_h)$$

The estimate $z_h$ can be viewed as a projection of the true error onto the subspace $W_h$. Note that this definition of the error estimate is global and its solution can be costly. Several solution methods will be discussed in Section 4.

Once $z_h$ is obtained, the reconstruction $R_h u_h$ can be defined as in (7). From (8), we have that if the saturation assumption (11) holds, we are able to control the finite element approximation error by minimizing the interpolation error of $z_h$.

Although the saturation assumption (11) has not been rigorously proven, some results on its validity for certain norms were provided recently. In [11], Dörfler and Nochetto have shown that small data oscillation implies the saturation assumption (11) in the $H^1_0$ seminorm.
for Poisson’s equation. Error estimates for the $L^2$ norm can be found in Dörfler [10]. In [1], a modified saturation assumption
\[ \| R_h u_h - u \| \leq \beta \| u_h - u \| + osc(F, T_h) \]
is introduced and proven to be valid for the energy norm, where the oscillation term $osc(F, T_h)$ depends on the regularity of the solution and the mesh. With this modification and the strengthened Cauchy-Schwarz-Bunyakovsky inequality, the following result holds for the hierarchical a posteriori error estimate $z_h$ (Achchab, Achchab and Agouzal [1], Theorem 2.1):
\[ ||| u - u_h ||| \leq \frac{C}{1 - \beta} ||| z_h ||| + \frac{1}{1 - \beta} osc(F, T_h). \]
This yields, instead of (8),
\[ ||| u - u_h ||| \leq \frac{C}{1 - \beta} IE(T_h, z_h) + \frac{1}{1 - \beta} osc(F, T_h). \]

If the solution $u$ is sufficiently regular or the mesh is appropriately refined, then $osc(F, T_h)$ is negligible and the saturation assumption holds, but on relatively coarse grids, in order to guarantee the convergence for the mesh adaptation process, $osc(F, T_h)$ should be controlled too. See [11] for a discussion on $osc(F, T_h)$.

In this paper, we construct optimal metric tensors with respect to interpolation error estimates $IE(T_h, z_h)$ for the $L^2$ norm. We assume that the reconstruction $R_h u_h = u_h + z_h$ defined in (7) gives a better approximation to $u$ than $u_h$, i.e., $\beta < 1$ in (1). Generally, due to the higher degree of approximation for the space $\tilde{V}_h$, one can expect that even $\beta = O(h^r)$, for some $r > 0$. In this case, $\beta \to 0$ as $h \to 0$, which is stronger than required by our assumption.

3. Metric tensor based on linear interpolation error estimate

3.1. Equidistribution and alignment. Let $\Omega$ be a polyhedral domain in $\mathbb{R}^d$ and let $T_h$ be a simplicial triangulation on $\Omega$. For every element $K \in T_h$, there exists an affine invertible mapping $F_K : \hat{K} \mapsto K$ such that $K = F_K(\hat{K})$, where $\hat{K}$ is the reference element. We assume that $\hat{K}$ has been chosen to be equilateral and have a unitary volume. We denote the Jacobian matrix of $F_K$ by $F'_K$ and the number of elements in $T_h$ by $N$.

As mentioned before, we consider an adaptive anisotropic mesh as a uniform mesh in the metric specified by a metric tensor $M$. Such a mesh is referred hereafter to as an $M$-uniform mesh. Following [18], it can be characterized by shape-orientation and size requirements on mesh elements.
Alignment condition (i.e., shape-orientation requirement). The elements of an $M$-uniform mesh $T_h$ are equilateral in the metric specified by $M$. This can be expressed as

$$
\frac{1}{d} \text{tr} \left( (F'_K)^T M_K F'_K \right) = \det \left( (F'_K)^T M_K F'_K \right)^{\frac{1}{2}}, \quad \forall K \in T_h
$$

where $M_K$ is an average of $M$ on element $K$.

Equidistribution condition (i.e., size requirement). The elements of an $M$-uniform mesh have an equal volume in the metric $M$, i.e.,

$$
|K| \sqrt{\det(M_K)} = \frac{\sigma_h}{N}, \quad \forall K \in T_h
$$

where

$$
\sigma_h = \sum_{K \in T_h} |K| \sqrt{\det(M_K)}.
$$

Note that the volume of an element $K$ in the metric $M_K$ is

$$
\int_K \sqrt{\det(M_K)} dx = |K| \sqrt{\det(M_K)}.
$$

3.2. Anisotropic interpolation error bound for piecewise quadratic functions. Elementwise anisotropic interpolation error estimates are developed in [19]. Consider the piecewise linear Lagrange interpolation ($k = 1$) of a piecewise quadratic function $v$ on an arbitrary mesh $T_h$. The elementwise interpolation error measured in the $L^q$ norm ($q \geq 1$) is given by

$$
\| v - I_h v \|_{L^q(K)}^q \leq C |K| \left( \text{tr} \left( (F'_K)^T |H_K| F'_K \right) \right)^q,
$$

where $H_K$ is the Hessian of $v$ on the element $K$, $|H_K| = \sqrt{H'_K H_K}$, $C$ is a constant independent of $T_h$ and $v$, and tr$(\cdot)$ denotes the trace of a matrix. Note that $H_K$ is constant on $K$ since by assumption $v$ is quadratic on the element. Summing over all elements of $T_h$ provides an upper bound for the global interpolation error

$$
\| v - I_h v \|_{L^q(\Omega)}^q \leq C \sum_{K \in T_h} |K| \left( \text{tr} \left( (F'_K)^T |H_K| F'_K \right) \right)^q.
$$

From this, we can set $IE(T_h, v)$ in (1) to

$$
IE(T_h, v) = \sum_{K \in T_h} |K| \left( \text{tr} \left( (F'_K)^T |H_K| F'_K \right) \right)^q.
$$
It has a lower bound as

\[
IE(T_h, v) = \sum_{K \in T_h} |K| \left( \frac{\text{tr} \left( (F'_K)^T |H_K| F'_K \right)}{\text{det} (|H_K|)} \right)^q 
\]

(14)

\[
\geq d^q \sum_{K \in T_h} |K| \left( \frac{\text{det} \left( (F'_K)^T |H_K| F'_K \right)}{\text{det} (|H_K|)} \right)^q 
\]

\[
= d^q \sum_{K \in T_h} |K| \left( \text{det} (|H_K|) \right)^{\frac{q}{d^q}} \left( \frac{\text{tr} (|H_K|)}{d^q} \right)^{\frac{q}{d^q}} 
\]

\[
= d^q \sum_{K \in T_h} \left( |K| \text{det} (|H_K|) \right)^{\frac{q}{d^q + 2q}} \left( \frac{\text{tr} (|H_K|)}{d^q} \right)^{\frac{q}{d^q + 2q}} 
\]

(15)

\[
\geq d^q N^{-\frac{2q}{d^q}} \left( \sum_{K \in T_h} |K| \text{det} (|H_K|) \right)^{\frac{q}{d^q + 2q}}. 
\]

If \( \max_{K \in T_h} \text{diam}(K) \to 0 \), where \( \text{diam}(K) \) denotes the diameter of \( K \), we see that the asymptotic lower bound on \( IE(T_h, v) \) is

\[
d^q N^{-\frac{2q}{d^q}} \left( \int_{\Omega} \text{det} (|H|) \frac{q}{d^q + 2q} \, dx \right)^{\frac{q}{d^q + 2q}}, 
\]

(16)

which is invariant for all meshes of the same number of elements \( N \). Thus, a mesh on which \( IE(T_h, v) \) attains a lower bound \( (15) \) can be considered to be an asymptotically optimal mesh.

3.3. Optimal metric. The optimal metric \( M \) is defined such that the interpolation error bound \( IE(T_h, v) \) defined in (13) attains its lower bound \( (15) \) on \( M \)-uniform meshes of \( N \) elements associated with \( M \).

We first notice that equality in (14) holds if the \( M \)-uniform mesh satisfies

\[
\frac{1}{d^q} \text{tr} \left( (F'_K)^T |H_K| F'_K \right) = \text{det} \left( (F'_K)^T |H_K| F'_K \right)^{\frac{1}{d^q}}, \quad \forall K \in T_h. 
\]

Comparing this with the alignment condition (10), a property satisfied by the \( M \)-uniform mesh, suggests that \( M \) be defined as

\[
M_K = \theta_K |H_K| 
\]

with some scalar function \( \theta_K \).

Next we notice that equality in (15) holds if the mesh satisfies

\[
|K| \text{det} (|H_K|)^{\frac{q}{d^q + 2q}} = \frac{1}{N} \sum_{K \in T_h} |K| \text{det} (|H_K|)^{\frac{q}{d^q + 2q}}, \quad \forall K \in T_h. 
\]

Comparing this to the equidistribution condition (11), another property satisfied by the \( M \)-uniform mesh, yields

\[
\sqrt{\text{det}(M_K)} = \text{det} (|H_K|)^{\frac{q}{d^q + 2q}}. 
\]
This condition can be used for determining $\theta_K$. Thus, we obtain the optimal metric tensor as

$$M_K = \det (|H_K|)^{-\frac{1}{d+2q}} |H_K|, \quad \forall K \in T_h. \tag{17}$$

The interpolation error bound (13) attains its lower bound (15) on any $M$-uniform mesh associated with this metric tensor. From (12) we obtain

$$\|v - I_h v\|_{L^q(\Omega)} \leq CN^{-\frac{d}{2}} \left( \sum_{K \in T_h} |K| \det(|H_K|)^{\frac{d}{d+2q}} \right)^{\frac{d+2q}{d}} \tag{18}$$

$$\sim CN^{-\frac{d}{2}} \left( \int_{\Omega} \det(|H|)^{\frac{d}{d+2q}} \, dx \right)^{\frac{d+2q}{d}}$$

for any $M$-uniform mesh associated with the metric tensor (17).

The metric tensor defined by (17) is not necessarily positive definite since both $|H_K|$ and $\det(|H_K|)$ can vanish locally. To avoid this problem, the error bound should be regularized with a positive parameter. The optimal metric tensor and corresponding interpolation error bound then read as

$$M_K = \det \left( I + \frac{1}{\alpha_h} |H_K| \right)^{-\frac{1}{d+2q}} \left( I + \frac{1}{\alpha_h} |H_K| \right), \quad \forall K \in T_h \tag{19}$$

$$\|v - I_h v\|_{L^q(\Omega)} \leq CN^{-\frac{d}{2}} \alpha_h \tag{20}$$

The regularization parameter $\alpha_h$ is defined through the equation

$$\sum_{K \in T_h} \sqrt{\det(M_K)} \, |K| = 2|\Omega|$$

or equivalently

$$\sum_{K \in T_h} \det \left( I + \frac{1}{\alpha_h} |H_K| \right)^{\frac{d}{d+2q}} \, |K| = 2|\Omega|. \tag{21}$$

It is easy to show that this equation has a unique solution. Moreover, it can be solved using a simple iteration scheme such as the bisection method. Furthermore, it can be shown that, when $dq \leq d + 2q$, $\alpha_h$ is bounded by

$$\left[ \frac{1}{2|\Omega|} \sum_{K \in T_h} \det (|H_K|)^{\frac{d}{d+2q}} |K| \right]^{\frac{d+2q}{dq}}$$

$$\leq \alpha_h \leq \left[ \frac{1}{d|\Omega|} \sum_{K \in T_h} (\text{tr}(|H_K|))^{\frac{dq}{d+2q}} |K| \right]^{\frac{d+2q}{dq}}.$$
4. Computation of the Metric Tensor and Anisotropic Meshes

We discuss here some implementation issues for two-dimensional problems.

The computation typically starts with a regular Delaunay mesh of the domain. For a given triangular mesh $T_h^{(i)}$ at step $i$, we compute the numerical approximation $u_h^{(i)}$ with a standard linear finite element method. Based on $u_h^{(i)}$ and $T_h^{(i)}$, we then compute $z_h^{(i)}$ as an approximation to the solution of the approximate error problem ($E_h$). Once $z_h^{(i)}$ has been obtained, it is straightforward to compute its elementwise Hessian and define the new metric tensor $M^{(i)}$ according to (19),

$$M_K^{(i)} = \det \left( I + \frac{1}{\alpha_h^{(i)}} |H_K(z_h^{(i)})| \right)^{-\frac{1}{q}} \left( I + \frac{1}{\alpha_h^{(i)}} |H_K(z_h^{(i)})| \right),$$

where the error is measured in the $L^2$-norm, i.e., $q = 2$. A new mesh is generated with the mesh generation software bamg (bidimensional anisotropic mesh generator) developed by F. Hecht [15] according to the metric tensor $M^{(i)}$. The process is repeated until a good adaptation is achieved.

A key component of the procedure is to find the solution $z_h$ of problem ($E_h$). Note that ($E_h$) is a global problem and finding its exact solution can be as costly as for computing a quadratic finite element approximation to the original PDE problem. Three approaches are considered here for solving or approximating ($E_h$).

Diagonalization. The expense of the error estimation can be significantly reduced, if the bilinear form $a$ in ($E_h$) is replaced by an approximation $\tilde{a}$ that allows a more efficient solution of the resulting linear system. A common and efficient approach is to replace the stiffness matrix by its diagonal. In two dimensions the estimation problem reduces to a series of local error problems which are defined over two elements sharing a common edge and can be solved efficiently. The approach is equivalent to the application of one Jacobi’s iteration (starting from zero) to the linear system resulting from the global error problem. This approach has been successfully used in finite element computations [7, 21, 22]. Moreover, it has been proven [3, 21] that such an error estimator is norm-equivalent to the original one under suitable conditions.

Despite its success in isotropic mesh adaptation, the approach does not seem to work well for anisotropic mesh adaptation. This may be explained by the fact that local error estimators generally depend on the aspect ratio of elements and can become inaccurate when the aspect ratio is large, a case that is often true for anisotropic meshes. Moreover, local estimators may not contain enough directional information.
of the solution which is global in nature and essential to the success of anisotropic mesh adaptation.

Node patch local error estimator. This approach is similar to the diagonalization approach, with the error estimator being obtained by solving a series of local error problems defined on node patches with homogeneous Dirichlet boundary conditions.

Inexact solution of the global error problem. In this approach the global error problem is kept but only an approximation to its exact solution is sought and used for the computation of the metric tensor. In our experiments, a few symmetric Gauss-Seidel iterations are employed to obtain such an approximation. It is noted that global error estimators have the advantages that they are often independent of element aspect ratio and contain more directional information of the solution. Moreover, it is known [8] that the global hierarchical basis error estimator is efficient and reliable for anisotropic meshes, provided that the strengthened Cauchy-Schwarz-Bunyakovsky inequality holds. In our case, this means that the maximum angle of triangular elements should be bounded away from 180°, which is a weak requirement on the mesh.

Numerical comparison among these approaches is given in the next section.

5. Numerical examples

In this section, we present some numerical results for a selection of two-dimensional problems with an anisotropic behaviour. We first compare different approaches in solving the error problem \( (E_h) \) and then the new method with some common Hessian recovery methods. At the end of the section, we give further examples to demonstrate the ability of the method to generate appropriate anisotropic meshes.

Convergence is illustrated by plotting the finite element solution error against the number of elements. We use the \( L^2 \)-norm for the error because the monitor function \( M_K \) is optimized for this norm.

5.1. A first example. Consider the boundary value problem

\[
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \\
u &= g \quad \text{on } \partial \Omega
\end{aligned}
\]

with \( \Omega = [0, 1] \times [0, 1] \). The right-hand side \( f \) and the Dirichlet boundary conditions are chosen such that the exact solution is given by

\[
u(x, y) = \tanh(60x) - \tanh(60(x - y) - 30) .
\]

The solution exhibits a strong anisotropic behaviour and describes the interaction between a boundary layer along the \( x \)-axis and a shock wave along the line \( y = x - 0.5 \). A solution plot is given in Fig. [a]
Local vs. global error estimators. As mentioned in Section 4, there can be a significant difference in accuracy between local and global error estimators on anisotropic meshes. In our first test, we investigate the influence of the three error estimators described in the previous section on mesh adaptivity.

Results for the error of the adaptive solution against the number of elements are presented in Fig. 2. As expected, the global error estimator works best, leading to a smaller error than those obtained with local error estimators. The local patch error estimator works better than the diagonalized one, mainly because it involves more elements and, in this sense, is more global.

The same observation can be made from Fig. 3 where adaptive meshes obtained with the error estimators are shown. All methods produce correct mesh concentrations, although mesh alignment and orientation are different. In the mesh controlled by the global error estimator elements near the boundary layer and the shock wave are very thin, have a large aspect ratio, and are properly aligned with the fronts of the shock wave and the boundary layer (Fig. 3c). On the other hand, the elements of meshes controlled by local error estimators have rather moderate aspect ratios and are, more or less, isotropic (Fig. 3a and 3b).

The accuracy of the corresponding finite element solutions is different, too. The mesh controlled by the global error estimator leads to a solution error $\|e\|_{L^2} = 2.2 \times 10^{-3}$, nearly one half of $\|e\|_{L^2} = 4.0 \times 10^{-3}$, the error obtained using the patch error estimator, and about one third of $\|e\|_{L^2} = 6.9 \times 10^{-3}$, the error achieved with the diagonalized error estimator.

These results are in good agreement with the comments made in Section 4 that the global error estimator will do a better job than local
Figure 3: Example 5.1 adaptive meshes obtained by means of the local and global a posteriori error estimators.
Anisotropic mesh adaptation based upon a posteriori error estimates

by means of a variational formulation [9]. Precisely, let

differentiating the polynomial twice.

Gauß-Seidel method produce an almost optimal mesh for this example.

error, but the difference is hardly visible. Three steps of the symmetric

see, a few iterations are sufficient for obtaining an approxima-

tion good

itera-

tions to compute the global error estimator. As one can

investigate how many iterations are sufficient for obtaining a va-

luable

Effect of the number of Gauß-Seidel iterations. We now

investigate how many iterations are suitable for obtaining a valuable

approximation to the error equation. Fig. 4 presents results for different

iteration numbers to compute the global error estimator. As one can

see, a few iterations are sufficient for obtaining an approximation good

enough for mesh adaptation. The convergence lines are very close to

each other. The exact solution of the error problem leads to a smaller

error, but the difference is hardly visible. Three steps of the symmetric

Gauß-Seidel method produce an almost optimal mesh for this example.

Comparison to Hessian recovery methods. Two Hessian re-

covery methods are considered for comparison purpose.

Quadratic least squares fitting. This method was recently developed

by Zhang and Naga [30] and proved to be robust and reliable. It com-

putes a local quadratic fitting to function values or their approxima-

tions at some neighboring points and obtains a Hessian approximation

by differentiating the polynomial twice.

Variational formulation. This approach recovers the Hessian, which

does not exist in the classical sense for piecewise linear functions,

by means of a variational formulation [9]. Precisely, let \( \phi_i \in V_h \)

be the piecewise linear basis function at node \((x_i, y_i)\). Then the nodal

ones for anisotropic mesh adaptation. Local error estimators are able

to capture the distribution of the magnitude of the true error and yield

a good mesh concentration. However, they fail to produce proper mesh

alignment, i.e., they do not contain enough information for proper

shape and orientation adaptation.

Figure 4: Example 5.1: a comparison of the error for adaptive finite

element solutions obtained with one and three symmetric Gauß-Seidel

iterations used in the solution of the linear system resulting from the

global error problem and with the exact solution of the system.
produce basically the same adaptive mesh. This seems to confirm the conjecture that highly accurate Hessian recovery is not necessarily more accurate and robust than the variational method.

The same approach is used to approximate $u_{xy}$ and $u_{yy}$.

Fig. 5 shows the error against the number of elements for each method. For comparison purpose, results obtained using the analytical Hessian are also included. All methods provide almost the same results. Particularly, the method based on the global estimator with three Gauss-Seidel iterations is comparable to the recovery-based methods.

It is worth noting that although the quadratic least squares fitting is generally more accurate and robust than the variational method, both produce basically the same adaptive mesh. This seems to confirm the conjecture that highly accurate Hessian recovery is not necessary for good mesh adaptation.

5.2. Further examples. We consider two boundary value problems in the form (22) with now the right-hand side $f$ and the Dirichlet boundary condition being chosen such that the exact solution is given by the following functions:

$$u_1(x, y) = \frac{1}{1 + e^{-0.5(x+y-1.25)}},$$

$$u_2(x, y) = e^{-25x} + e^{-25y}.$$  

The first function represents a shock wave along the line $y = 1.25 - x$ while the second models a boundary layer near the coordinate axes.

We compare the error for finite element solutions obtained with the global error estimator and the quadratic least squares Hessian recovery.
Anisotropic mesh adaptation based upon a posteriori error estimates

(a) A comparison of the error for adaptive finite element solutions.

(b) Diagonalized error estimator: adaptive mesh with 618 vertices and 1153 triangles (left); a close-up view near (0.7, 0.7) (right); $\|e\|_{L^2} = 1.5 \times 10^{-3}$.

(c) Quadratic least squares Hessian recovery: adaptive mesh with 620 vertices and 1139 triangles (left); a close-up view near (0.7, 0.7) (right); $\|e\|_{L^2} = 3.2 \times 10^{-4}$.

(d) Global error estimator: adaptive mesh with 610 vertices and 1119 triangles (left); a close-up view near (0.7, 0.7) (right); $\|e\|_{L^2} = 3.8 \times 10^{-4}$.

Figure 6: BVP \(^{22}\) with the exact solution $u(x, y) = 1/(1 + e^{x+y-1.25})$: adaptive meshes and close-up views near (0.7, 0.7).
(a) A comparison of the error for adaptive finite element solutions.

(b) Diagonalized error estimator: adaptive mesh with 622 vertices and 1155 triangles (left); a close-up view near (0.2, 0.1) (right); $\|e\|_{L^2} = 3.1 \times 10^{-3}$.

(c) Quadratic least squares Hessian recovery: adaptive mesh with 616 vertices and 1146 triangles (left); a close-up view near (0.2, 0.1) (right); $\|e\|_{L^2} = 1.4 \times 10^{-3}$.

(d) Global error estimator: adaptive mesh with 616 vertices and 1147 triangles (left); a close-up view near (0.2, 0.1) (right); $\|e\|_{L^2} = 1.4 \times 10^{-3}$.

Figure 7: BVP (22) with the exact solution $u(x, y) = e^{-25x} + e^{-25y}$, adaptive meshes and close-up views near (0.2, 0.1).
Results for the quasi-uniform (regular Delaunay) mesh and the local error estimator are also given. Fig. 6 and 7 show the results.

As in 5.1, we can see that mesh adaptation significantly reduces the finite element error compared to a quasi-uniform mesh having the same number of elements. The mesh based on the local error estimator provides a good mesh concentration and is clearly better than a quasi-uniform one, but it is almost isotropic and inferior to a mesh obtained with the use of the global error estimator. Again, one can observe that the elements of the meshes obtained by means of the global error estimator and the quadratic least squares fitting are properly aligned with the shock wave and the boundary layers. The new method produces results comparable to those obtained with recovery-based methods.

5.3. **Discontinuous gradients.** Next, we consider problems whose solution has a discontinuous gradient along a certain interface in the domain. This situation arises in elliptic problems with discontinuous coefficients in the diffusion term such as heat conduction problems with jumps in material coefficients. Difficulties when using gradient recovery methods for such problems were already pointed out in [23], and this is true for the Hessian recovery as well: if the numerical approximation is accurate enough, we should expect a discontinuity in its gradient and its Hessian. Since most Hessian recovery methods employ some sort of averaging over a certain region, they can be very inaccurate near discontinuities. This issue can readily be observed in the following simple example.

Let $\Omega = [0, 1] \times [0, 1]$. Consider the boundary value problem

$$
-\alpha \Delta u = 0 \quad \text{in } \Omega \\
u = g \quad \text{on } \partial \Omega,
$$

where

$$
\alpha = \begin{cases} 
1, & x < 0.5 \\
\alpha, & x \geq 0.5
\end{cases}
$$

and the Dirichlet boundary condition is chosen such that the exact solution is given by

$$
u(x, y) = \begin{cases} 
-2\alpha x + \alpha + 1, & x < 0.5 \\
-2x + 2, & x \geq 0.5.
\end{cases}
$$

The solution has a gradient jump of magnitude $\alpha$ across the line $x = 0.5$, but is continuous on $\Omega$ and linear in each of the subdomains. We take $\alpha = 6$ in our computation.

We first consider the situation where the mesh does not contain the information of the interface. In this situation at least part of the interface does not consist of edges. In order to match the sharp bend in the solution along the interface, the adaptive mesh should exhibit a strong concentration of elements around $x = 0.5$ oriented along the
(a) Finite element solutions for the problem \( \text{Example 5.3} \) without the interface being present in the mesh, the solution is not exact (left); with the interface being present in the mesh, the solution is exact (right).

(b) Adaptation without predefined interface edges: quadratic least squares Hessian recovery, 442 vertices, \( \| e \|_{L^2} = 4.2 \times 10^{-3} \) (left); global error estimator, 429 vertices, \( \| e \|_{L^2} = 4.3 \times 10^{-3} \) (right).

(c) Adaptation with predefined interface edges: quadratic least square Hessian recovery: 105 vertices, \( \| e \|_{L^2} = 3.9 \times 10^{-16} \) (left); global error estimator, 101 vertices, \( \| e \|_{L^2} = 3.9 \times 10^{-16} \) (right).

Figure 8: Example \( \text{5.3} \) gradient jump along the line \( x = 0.5 \). Adaptive meshes and finite element solutions with and without the predefined interface edges.
interface. In this test, the quadratic least squares and the global hierarchical basis error estimator both succeed in providing an appropriate mesh adaptation and, again, deliver comparable results (Fig. 8b).

The situation is different if the interface is present in the mesh. In this case, the analytical solution $u$ belongs to the corresponding finite element space and, consequently, the numerical approximation computed by means of the linear finite element method is exact (Fig. 8a, right). Hence, no adaptation is required and the proper mesh should be a uniform mesh. Now, consider the mesh adaptation using the quadratic least squares Hessian recovery. Because of the sharp bend in the solution, the recovered Hessian should be very large, $\approx O(1/h)$, near $x = 0.5$, but zero elsewhere, because the solution is linear in each of the subdomains. This should lead to an excessive over-adaptation near the interface. On the other hand, we expect no adaptation for the hierarchical basis error estimator in this case because the numerical solution is exact and, consequently, the error estimator is zero everywhere in $\Omega$. A quasi-uniform mesh should result. Fig. 8c presents mesh examples. We see that the adaptation by means of the Hessian recovery (left) leads to a strong element concentration along the interface line, as predicted, whereas the mesh based on the hierarchical error estimator (right) is almost uniform.

We also expect a similar behaviour of these methods for general problems exhibiting gradient jumps or similar discontinuities along internal interfaces. Thus, for such problems, it can be of advantage to use the a posteriori error estimator for effective mesh adaptation because of the more efficient employment of given degrees of freedom.

6. Heat conduction in a thermal battery

In this section we consider heat conduction in a thermal battery with large orthotropic jumps in the material coefficients. The mathematical model considered here is taken from [23, 25] and described by

\begin{equation}
\begin{aligned}
\nabla \cdot (D^k \nabla u) &= f^k & \text{in } \Omega \\
D^k \nabla u \cdot n &= g^i - \alpha^i u & \text{on } \partial \Omega,
\end{aligned}
\end{equation}

where $\Omega = [0, 8.4] \times [0, 24]$, and

$$D^k = \begin{bmatrix}
D^k_x & 0 \\
0 & D^k_y
\end{bmatrix}.$$

The data for each material $k$ and for each of the four sides $i$ of the boundary starting with the left hand side boundary and ordering them clockwise are given in Table 1.

The analytical solution for this problem is unavailable. The geometry and the contour and surface plots of a finite element approximation are given in Fig. 9.
<table>
<thead>
<tr>
<th>Region $k$</th>
<th>$D_x^k$</th>
<th>$D_y^k$</th>
<th>$f^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td>25</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>0.8</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>0.0001</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
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<td>0</td>
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<tr>
<td>5</td>
<td>0.05</td>
<td>0.05</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) Material coefficients.

<table>
<thead>
<tr>
<th>Boundary $i$</th>
<th>$\alpha^i$</th>
<th>$\varphi^i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</tr>
<tr>
<td>2</td>
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</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

(b) Boundary conditions.

Table 1: Heat conduction in a thermal battery: material coefficients and boundary conditions.

Figure 9: Heat conduction in a thermal battery: (a) device geometry, (b) contour plot, and (c) surface plot of a linear finite element solution.

We compare the quadratic least squares Hessian recovery and the global error estimator. For this example we found that three steps of the symmetrical Gauß-Seidel method were not sufficient for a full mesh adaptation and increased the number to seven, which proved to
Figure 10: Heat conduction in a thermal battery: a comparison of the error for adaptive finite element solutions obtained on meshes (a) with and (b) without the interfaces being present in the mesh.

be enough to achieve at least a comparable error estimate as the one obtained with quadratic least squares Hessian recovery.

Fig. 10 shows global error estimates (obtained by solving exactly the approximate error problem ($E_h$)) for finite element solutions on adaptive meshes controlled by the global error estimate or Hessian recovery and having all or no predefined interface edges. (The interface consists of edges when a mesh has all predefined interface edges.)

Typical adaptive meshes with predefined interface edges for both methods are shown in Fig. 11.

The results are in good agreement with those in Section 5.3. When the interface edges are not present in the mesh, both methods provide similar results. On the other hand, when the mesh contains all the information of the interface, the quadratic least squares Hessian recovery produces a mesh with strong element concentration near all internal interfaces (Fig. 11a), whereas the global error estimator leads to a mesh (cf. Fig. 11b) that has higher element concentration in the
(a) Quadratic least squares Hessian recovery, 3511 vertices.
(b) Global error estimator, 3513 vertices.

Figure 11: Heat conduction in a thermal battery: adaptive meshes obtained with (a) quadratic least squares Hessian recovery and (b) global error estimator.

corners of the regions, has a proper element orientation near the interfaces between the regions 2 and 3, and is almost uniform in regions where the solution is nearly linear (cf. Fig. 9c for the surface plot of a computed solution).

Once again, the numerical results for this example show that a recovery method can lead to over-concentration of elements. The new
Anisotropic mesh adaptation based upon a posteriori error estimates

method, on the other hand, produces only necessary concentration and is also able to catch the directional information of the solution required for proper element alignment. This example also demonstrates that the new method can be successfully used for problems with jumping coefficients and strong anisotropic features.

7. Conclusions and Comments

In the previous sections we have presented a mesh adaptation method based on hierarchical basis error estimates and shown that anisotropic mesh adaptation can be successfully controlled by a posteriori error estimators. Numerical results have shown that the new method is comparable in accuracy with commonly used Hessian-recovery-based methods and can be more efficient for some examples by producing only necessary element concentration.

A key idea in the new approach is the use of the global hierarchical error estimator for reliable directional information of the solution. To avoid the expensive exact solution of the global error problem, we employed only a few steps of the symmetric Gauß-Seidel iteration for the efficient solution of the resulting linear system. Numerical results have shown that this is sufficient for obtaining an approximation to the error good enough for the purpose of mesh adaptation.

The theoretical validation of the new approach is upon the satisfaction of the saturation assumption and the strengthened Cauchy-Schwarz-Bunyakovskv inequality. The former can be assumed as valid if the analytical solution is smooth enough or the mesh is appropriately refined. The latter is valid as long as the largest interior angle of the triangulation is bounded away from 180°. It appears that in many cases both of these assumptions can be fulfilled in practice.

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